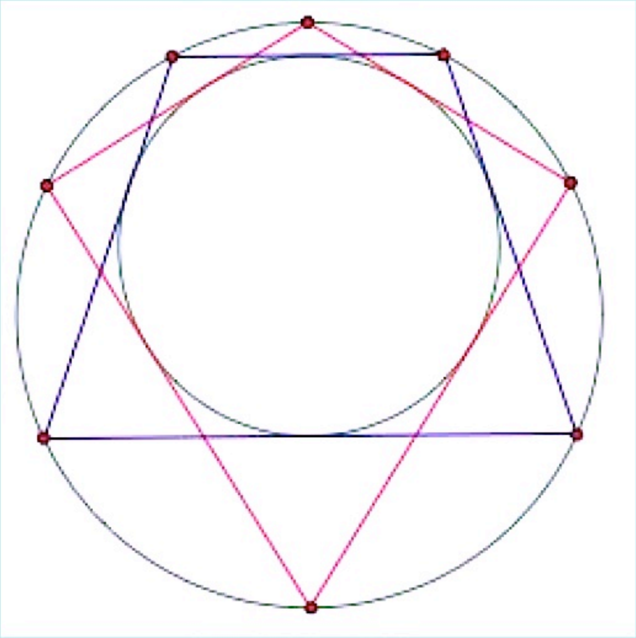
Some Adventures in Euclidean Geometry



Michael de Villiers

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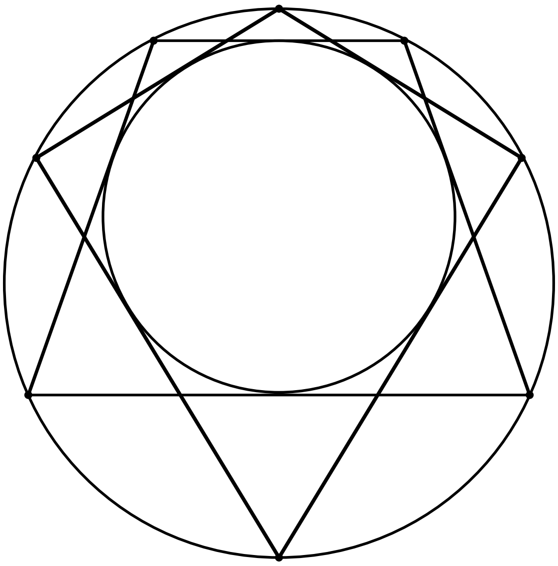

Michael de Villiers

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"Life without geometry is pointless."

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Preface

> "*... the spirit of mathematics ... is active rather than contemplative - a spirit of disciplined search for* ***adventures*** *of the intellect.*" - Alfred Adler (1984:7)

To the author also, mathematics is an exciting never-ending adventure, always full of lovely and beautiful surprises. It is like wandering in an uncharted jungle where one never knows what sparkling brooks, cascading waterfalls, exotic plants and strange animals may lurk just around the next corner. If one is prepared to keep an open mind, asking questions and continually exploring, mathematics provides an inexhaustible source of inspiration and stimulation; there is always something new to discover, or at the very least, new ways of looking at old results. Of course, like in any jungle, there is also danger in various forms, and one has to constantly guard in one's explorations against false conclusions and conjectures.

Unfortunately many people seem to regard mathematics in general as a boring, dead subject with nothing new to discover. In particular with regard to geometry, they believe that the old Greeks and other ancient civilizations have already discovered all there is to discover in geometry about 2000 years ago. This is however not the case; many interesting and beautiful geometric results have been discovered during the past 300 years. Apart from dealing with some such examples like the Euler line, the theorems of Ceva, Napoleon, Morley, Miquel, Varignon, etc., this book will also present some generalizations of these, and other results which, as far as the author knows, are original and have not been published elsewhere before. Extensive attention is also given to the classification of the quadrilaterals from the symmetry of a *side-angle* duality.

This book does not follow a traditional mathematics textbook approach by starting from carefully defined axioms and definitions, and monotonously churning out one after the other, Theorem 1, Theorem 2, etc. Instead, this book attempts to actively involve the reader in the heuristic processes of conjecturing, discovering, formulating, classifying, defining, refuting, proving, etc. Mathematics is not a spectator, but a participator discipline; one simply cannot sit on the sideline and watch other's play; one must get involved to appreciate and enjoy it. The reader should therefore preferably always have paper and pencil handy. It should also be noted that later chapters build on the preceding exercises; so it is advisable to work through the chapters and exercises in sequence.

Exploration on computer by construction and measurement with *Sketchpad*, or other dynamic geometry programs like *Cabri*, *Cinderella*, etc., is strongly encouraged throughout, although not essential. Deductive proof is furthermore not presented here only as a means of verification, but also as a means of explanation, further discovery and systematization.

It is further assumed that the reader is at least acquainted with high school geometry and trigonometry (e.g. congruency, similarity, circles, concurrency, sine & cosine rules, etc.) Some of the exercises would however be accessible to junior secondary students as well. Some references for further background reading are provided in the bibliography. Most of the content comes out of the author's on-and-off explorations and experiences in geometry over the past fifteen years or so. The book starts in the style of Lakatos' **Proofs and Refutations** with the fictional dramatization of an actual classroom episode some years back. The reader is then led through various explorations and extensions of high school geometry. The book is therefore aimed mainly at gifted high school pupils, secondary mathematics teachers who are looking for enrichment material and the undergraduate training of prospective mathematics teachers.

Note

Although a list of quadrilateral definitions are provided at the back for referencing purposes, it is expected of readers to formulate their own definitions for the various quadrilaterals as they go along, and to keep and update their own lists.

Also view a Hierarchical Classification of Quadrilaterals at:
http://dynamicmathematicslearning.com/quad-tree-new-web.html

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Geometer's Sketchpad and Activity Modules are free to download at:
http://dynamicmathematicslearning.com/free-download-sketchpad.html

* *Sketchpad* is available at $70 (single), $250 (10 lab pack), $1000 (50-User site license) or $1500 (Unlimited User site license) from **Key Curriculum Press**, Key Curriculum Press, 1150 65th Street, Emeryville, CA 94608, USA. Phone US: Monday through Friday from 6 a.m. to 5 p.m. PT 800-995-MATH (6284). http://www.keypress.com/x1571.xml

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Contents

Chapter 1

A classroom episode

"A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest, but if it challenges your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."
- George Polya (1973 : v)

A teacher is busy revising cyclic quadrilaterals when Peter suddenly asks if there are any cyclic quadrilaterals that are also kites, and if there are, what their properties would be.

TEACHER: "That is an excellent question, Peter! Your question is of course part of the more general question of where the cyclic quadrilaterals fit into the classification of the quadrilaterals. To do that, we first need to revise our classification scheme for quadrilaterals from Std. 7. Who can still remember it?

JAN: "I can, sir."

TEACHER: "Okay Jan, please come and draw your scheme on the blackboard."

Jan walks to the blackboard and draws the scheme shown in Figure 1.

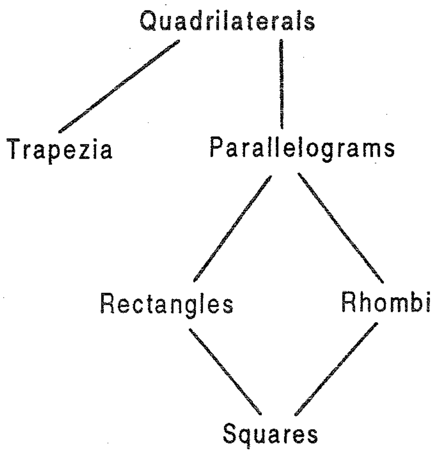


Figure 1

TEACHER: "Well, class, what do you think? Do you all agree with his classification?"

SUSAN: "No, sir. I would have classified the parallelograms beneath the trapezia. And he's also left out the kites completely."

TEACHER: "Now Susan, why do you say that the parallelograms should be classified beneath the trapezia?"

SUSAN: "Well, isn't the definition of a trapezium that it is a quadrilateral with *at least* one pair of opposite sides parallel, and that surely includes a parallelogram which has two pairs of opposite sides parallel?"

TEACHER: "I see. Jan, what do you think? Do you agree with her definition of a trapezium?"

JAN: "No, sir. Is a trapezium not a quadrilateral with *only* one pair of opposite sides parallel? I seem to recall that's what our Std. 7 teacher taught us."

TEACHER: "Yes and no, Jan. You see, both your and Susan's definitions are correct, in the sense, that both give a correct description of a trapezium. And since our choice of definition is to a certain extent arbitrary, we could choose either of your definitions. However, Susan's has the advantage of allowing us to classify the parallelograms beneath the trapezia. Parallelograms can then be considered as special kinds of trapezia. For economical reasons it is useful to define quadrilaterals in terms of other quadrilaterals. For example, if we define a square as a special rectangle, then all the rectangle theorems (e.g. equal diagonals) automatically apply to squares, and there is no need to prove it again for squares. In mathematics we therefore generally prefer defining and classifying geometric figures hierachically, that is, defining them as subsets of more general figures in such a way that the relationships between the figures are highlighted. Since it would furthermore be akward having two different definitions, could we agree on Susan's definition just for the sake of having consensus?" (Class nods).

"But Susan also raised another point, namely, the omission of the kites from the scheme. Who can still remember what a kite is?"

Several hands are immediately raised.

TEACHER: "Okay, Peter, come to the board and draw a kite."

Peter walks to the board and draws the sketch shown in Figure 2.

TEACHER: "Good. Where would you place the kites in our earlier classification scheme? Please fill it in on Jan's sketch."

Peter redraws Jan's diagram as shown in Figure 3.

TEACHER: "Well, class, what do you think? Do you agree? ... Anne?"

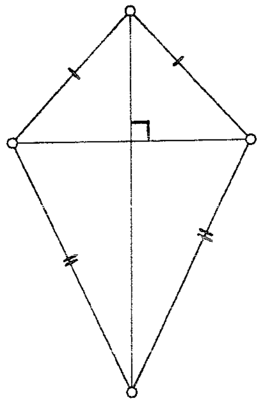


Figure 2

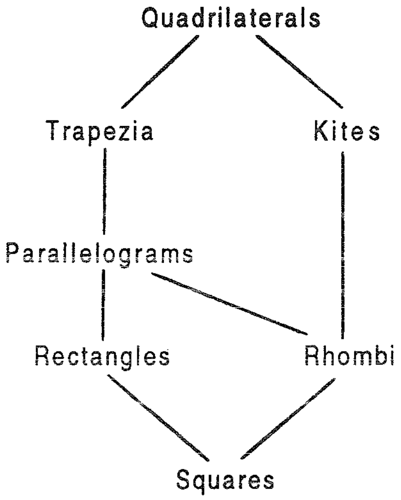


Figure 3

ANNE: "Sir, I cannot remember my Std. 7 work so well, but a square or rhombus doesn't look like a kite to me".

TEACHER: "Peter, would you please explain to Anne why you say a square and a rhombus are kites?"

PETER: "Well, a rhombus or square has *all* the characteristic properties of a kite, namely, two pairs of adjacent sides equal, perpendicular diagonals and an axis of symmetry through one pair of opposite angles. Of course, rhombi and squares are *special* kites since they have *additional* properties, for example, all sides equal, two axes of symmetry through both pairs of opposite angles, etc."

TEACHER: "Do you now understand, Anne?"

ANNE: "Yes, sir. Now I remember."

TEACHER: "Okay, that brings us to Peter's original question of where the cyclic quadrilaterals fit into this classification scheme. Before we try to do that, who can give us a definition for cyclic quadrilaterals? ... Yes, Susan?"

SUSAN: "A cyclic quadrilateral is a quadrilateral of which the vertices fit on a circle?"

TEACHER: "Good, Susan. Who can mention some more properties of cyclic quadrilaterals? ... Jan?"

JAN: "Well, sir, their opposite angles are supplementary, and their exterior angles are equal to the opposite interior angles."

TEACHER: "Quite correct. You will remember that we just proved it the other day. Which of the quadrilaterals in this diagram will fit on a circle or has the properties Jan has mentioned? ... Yes, Safa?"

SAFA: "Oh, I know sir! The rectangles and squares! Their angles are all 90 degrees and therefore opposite angles are supplementary, and they are therefore cyclic."

TEACHER: "That's quite correct, Safa! Where would one have to place the compass to construct the circumscribed circle of a rectangle or square? ... Jan?"

JAN: "It's easy, sir. At the intersection of the diagonals."

TEACHER: "Why do you say so, Jan?"

JAN: "Their diagonals are equal and bisect each other in equal parts. By therefore placing the point of the compass at the intersection of the diagonals, and opening the other point to a vertex, the circumscribed circle can easily be drawn."

ANNE: "But what about a parallelogram? Is it not also possible to draw a circle around it by placing the compass at the intersection of its diagonals since they also bisect each other?"

Teacher keeps quiet and looks at Jan.

JAN: "No, that won't be possible since the distances from the point of intersection of the diagonals of a parallelogram to its vertices are not constant since the diagonals are not equal." (walks to the board and illustrates this).

ANNE: "Oh yes, now I see it ..."

SAFA: "Another way of seeing it is ... the opposite angles of a parallelogram are equal, and therefore for them to be supplementary, the opposite angles would have to be both 90 degrees ... therefore only rectangles or squares."

TEACHER: "An excellent explanation, Safa. Let's now draw in the cyclic quadrilaterals on our diagram."

Teacher walks to the board and redraws the diagram as shown in Figure 4.

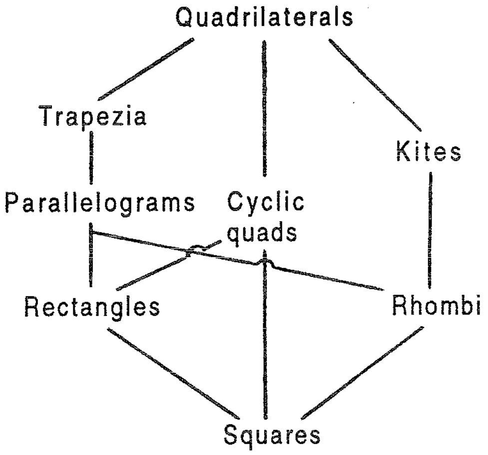


Figure 4

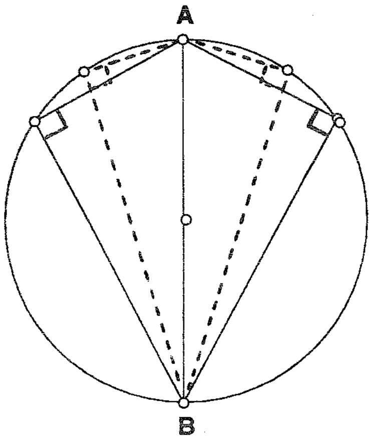


Figure 5

TEACHER: "In the diagram we can now see that the intersection of the parallelograms (parallel sides) and the kites (two pairs of adjacent sides equal) are the rhombi; in other words, the rhombi have all the properties of the parallelograms *combined* with that of the kites. Similarly the squares have the combined properties of the rectangles and rhombi (eg. all angles and sides equal). As the diagram is now drawn it suggests that there is no intersection between the cyclic quadrilaterals and kites. This brings us to Peter's question. Who can think of an example of a quadrilateral, apart from a square, that has all the properties of the cyclic quadrilaterals and kites?"

After a while:

PETER: "The following figures, sir! Can I come and draw it on the board?"

Teacher nods, and Peter draws the sketch shown in Figure 5.

TEACHER: "That's very nice, Peter. Note that AB is a diameter and that the angles thus formed in the semi-circles are 90°. And what is a possible definition for this class of figures? ... Anne?"

ANNE: "Those kites with *only* one pair of opposite angles equal to 90°."

SUSAN: "No, sir, I do not agree. A hierarchical definition would be to define them as those kites with at least one pair of opposite angles equal to 90°, so that we can consider the squares as special cases."

SAFA: "I agree with Susan, but what bothers me is whether these figures include all the possible cases for the intersection between the cyclic quadrilaterals and kites. Is it always the case that at least one pair of opposite angles are 90°? Are there not perhaps other possibilities?

TEACHER: "A good question, Safa. Well, class, what do you think? Can we prove that a kite that is also cyclic must have at least one pair of opposite angles each equal to 90°?"

JAN: "But, sir, it's obvious from the sketch. If the one pair of opposite angles are not 90°, it would not be cyclic."

PETER: "Well, why don't we define them as kites with opposite angles supplementary? Since at least one pair of opposite angles of a kite are equal (we proved that in Std. 8, I think), the requirement that they must be supplementary immediately implies that there must be at least one pair of opposite right angles."

TEACHER: "That's a good suggestion, Peter. I think let's stick to it for the moment. In any case, who wants to suggest a suitable name for these figures we have discovered? ... Yes, Anne?"

ANNE: "Let's call them *right kites*, since they have right angles."

TEACHER: "Good, that's not a bad name. Any other suggestions? ... No? ... Okay, then let's go on. What about the possibility of an intersection between the trapezia and cyclic quadrilaterals? In other words, are there any figures which possess the combined properties of both, e.g. the parallelness of at least one pair of opposite angles, as well as the supplementarity of opposite angles?"

After a while:

SUSAN: "Yes, sir! *Isosceles trapezia*! I remember we had such figures in Std. 8."

TEACHER: "Excellent, Susan! Please come and draw an example on the board and explain why you say it is cyclic."

Susan walks to the blackboard and draws the sketch shown in Figure 6.

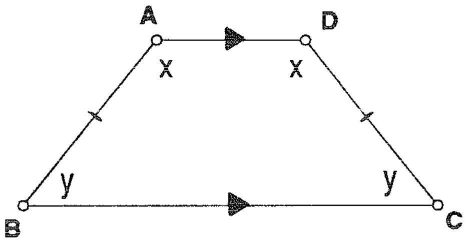


Figure 6

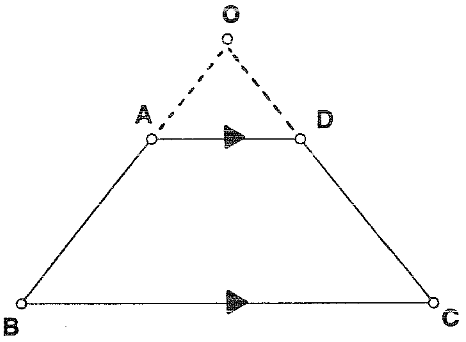


Figure 7

SUSAN: "I remember it has at least one pair of opposite sides parallel and the other pair equal, as well as two pairs of adjacent angles equal. If we let $\angle B = y = \angle C$ and $\angle A = x = \angle D$, then since AD//BC we have $\angle A + \angle B = 180°$. In other words, $x + y = 180°$, and therefore $\angle A + \angle C = 180°$. Thus, the isosceles trapezium is cyclic, since the opposite angles are supplementary."

TEACHER: "Well done, Susan. But how can we be sure that the isosceles trapezia are the only intersection possible between the trapezia and cyclic quadrilaterals? Are there not other possibilities?"

After a while:

PETER: "No, sir, there aren't any other possibilities. For example, (Peter walks to the board and draws the sketch shown in Figure 7) we can easily prove as follows that if ABCD is cyclic with AD//BC, then AB=DC, $\angle A = \angle D$ and $\angle B = \angle C$; in other words, that ABCD is an isosceles trapezium. For instance, $\angle A + \angle C = 180°$ since ABCD is cyclic, but $\angle A + \angle B = 180°$ since AD//BC. Therefore $\angle B = \angle C$ and therefore also $\angle A = \angle D$. Since angles OAD and ODA are the supplements of the equal angles A and D, it therefore also follows that $\angle OAD = \angle ODA$. Therefore triangles OAD and OBC are both isosceles and we have OB-OA = OC-OD and consequently AB=DC."

TEACHER: "Very good, Peter. Who would now like to suggest a definition for isosceles trapezia?" ... Yes, Safa?"

SAFA: "An isosceles trapezium is a quadrilateral with at least one pair of opposite sides parallel and at least one pair of opposite sides equal."

ANNE: "No, sir, then she includes the parallelograms, for example ..."

Anne walks to the board and draws the sketches shown in Figure 8.

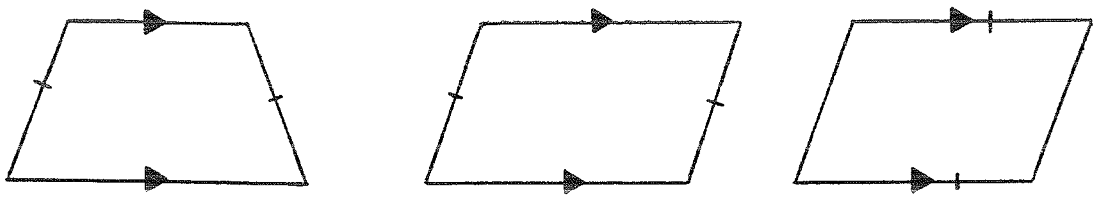


Figure 8

ANNE: "All these figures are included by her definition, but the last two are parallelograms which are *not* cyclic. Her definition can rather be changed to those quadrilaterals with at least one pair of opposite sides parallel, and at least the other pair equal, but not parallel. Then it does not include the parallelograms."

SUSAN: "I don't like Anne's definition - it's too long and actually excludes the squares and rectangles, which are also cyclic even though they are parallelograms. I would like to suggest that we define isosceles trapezia as those cyclic quadrilaterals with at least one pair of opposite sides parallel, or alternatively as those trapeziums with opposite angles supplementary. We would then clearly exclude the general parallelograms, but not the rectangles and squares. Furthermore, as Peter has shown, we can easily deduce its other properties from this definition."

TEACHER: "How do you feel, class? Shall we accept Susan's suggestion as our definition? ... Yes? Okay, let me then draw in the right kites and isosceles trapezia on our classification diagram".

Teacher walks to the board and redraws the diagram as shown in Figure 9.

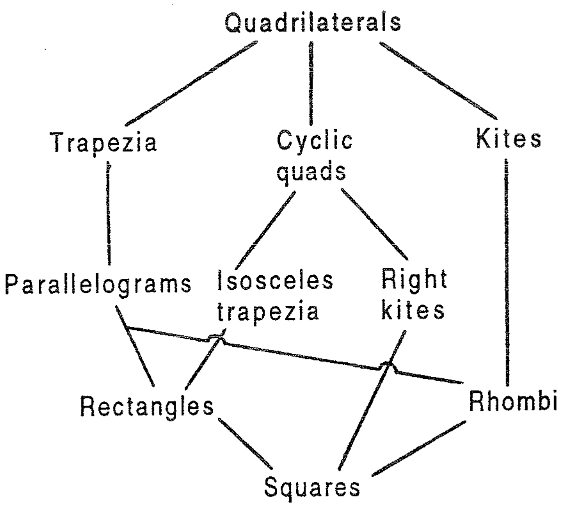


Figure 9

ANNE: "But, sir, why did you not connect the isosceles trapezia with the squares and the right kites with the rhombi?"

TEACHER: "What do you think, class, should I connect them?"

SAFA: "No, sir, it's not necessary to connect the isosceles trapezia with the squares as it is clear from the diagram that the rectangles are special cases, and therefore also the squares ... But I don't know about the connection between the right kites and rhombi"

JAN: "I've just got the answer to that. We should not connect them since a right kite with all its sides equal is the same as a rhombus with a right angle, which is a square. There are therefore no rhombi, other than the squares, that are right kites."

TEACHER: "That's quite correct. Let's now look at the diagram overall. Firstly, we notice that from the top to the bottom the figures, and their properties, become more and more specialized. For example, the trapezia need only have one pair of opposite sides parallel, but the parallelograms have two pairs of opposite sides parallel. The rectangles then obtain the additional property of the supplementarity of opposite angles from the isosceles trapezia, while the squares obtain the equality of sides from the rhombi. The same thing can be observed for other properties, such as equal or perpendicular diagonals, axes of symmetry, equal opposite angles, etc. For instance, a trapezium in general does not have equal diagonals, but the isosceles trapezia have. Therefore the intersection of the isosceles trapezia and parallelograms gives us the rectangles with equal (bisecting) diagonals."

"Similarly the figures become more general as we move up from the bottom to the top. For example, a rhombus is more general than a square, and a kite in turn than a rhombus. If one now looks at a usual definition for kites, namely a quadrilateral with two pairs of adjacent sides equal, one may wonder if it is possible to further generalize the concept kite. Who of you would like to suggest some of the conditions for a kite we could leave out in order to define a new figure of which the kites are a special case?"

SAFA: "What about a quadrilateral with at least one pair of adjacent sides equal?"

TEACHER: "Very good, Safa. Let us call figures with these properties *skew kites*. The question is however how the inclusion of these figures will influence our classification scheme. What in particular should we also consider? ... Yes, Susan?"

SUSAN: "We will have to look carefully at the possibility of new intersections between the skew kites and the cyclic quadrilaterals and trapezia and try and accommodate it in our diagram."

TEACHER: "Quite correct, Susan. How could we define a figure that has all the combined properties of a trapezium and a skew kite? ... Yes, Safa?"

SAFA: "What about: those skew kites with at least one pair of opposite sides parallel?"

TEACHER: "That's nice. Could you please draw us some examples on the board?"

Safa walks to the blackboard and draws the figures shown in Figure 10.

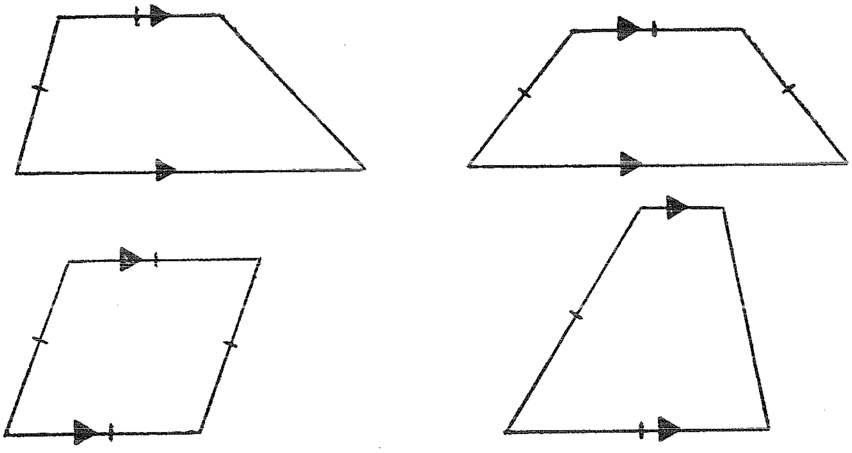


Figure 10

TEACHER: "Good. Now class, what do you think? Did she cover all possible examples? ... Yes, Anne?"

ANNE: "No, sir, she did not. She left out the squares, which is also an example of these figures."

JAN: "And the last one and the first one are basically the same, sir."

TEACHER: "Okay, what do you think is a suitable name for these figures? ... Yes, Anne?"

ANNE: "Let's call them *skew trapezia*, sir."

The rest of the class nods in confirmation.

TEACHER: "Well it looks as if we have consensus with this name. Let us go a bit further with the skew trapezia. What would you say is the intersection between the skew trapezia and the kites, if any? .. Yes, Jan?"

JAN: "One can easily see from Safa's figures that it must be a rhombus, sir."

TEACHER: "Are you really sure? ... Not quite sure it seems. Okay, for homework I want you to verify if a kite with at least one pair of opposite sides parallel, is a rhombus."
"In any case, what about the right kites? What is the intersection between them and the skew trapezia? ... Yes, Safa?"

SAFA: "The squares, sir?"

TEACHER: "Good, and between the skew trapezia and isosceles trapezia? ... Yes, Peter?"

PETER: "Safa's second figure, let's say a *trilateral trapezium*, sir."

TEACHER: "Yes, and of course the squares also as special cases. Who wants to try and formulate a definition for the trilateral trapezia? ... Yes, Jan?"

JAN: "Let's define them as isosceles trapezia with at least three sides equal, since it would include the squares but not the rhombi as the rhombi are not isosceles trapezia."

PETER: "Why don't we simply say it is a trapezium with at least three sides equal?"

SUSAN: "That won't be correct, as it would then include the rhombi. Remember that our definition for the trilateral trapezia must incorporate all the properties of the isosceles trapezia, as well as those of the skew trapezia. My suggestion for an alternative to Jan's definition is: a cyclic quadrilateral with at least three sides equal."

SAFA: "Susan, does your definition really supply enough information to for example prove that at least one pair of opposite sides are parallel?"

SUSAN: "I think so - it seems logical, but I can't think of a proof immediately."

TEACHER: "Susan, why don't you come to the board and try and prove it?"

SUSAN: "Well, I can try, sir."

She walks to the board and draws the diagram shown in Figure 11.
She thinks in silence for a while.

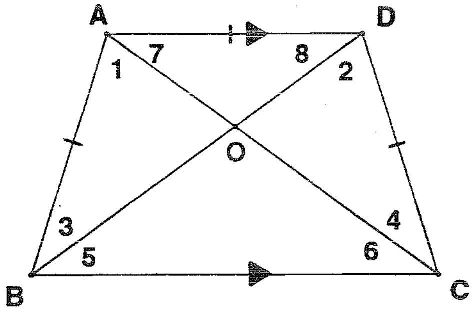


Figure 11

SUSAN: "Oh, it's easy. Since ABCD is cyclic, we have $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ because they lie on the same chords. Therefore, triangles OAB and ODC $(\angle,\angle,s)$ are congruent which implies OB=OC. This implies $\angle 5 = \angle 6$, but $\angle 5 = \angle 7$ since they lie on the same chord DC. Therefore $\angle 6 =$ alternate $\angle 7$ which implies AD//BC."

SAFA: "Oh, now I see it ... But you didn't use the fact that AD is equal to AB and DC... in other words we could earlier also have defined an isosceles trapezium as a cyclic quadrilateral with at least one pair of opposite sides equal, and done the same proof to show that at least one pair of opposite sides are parallel, not so, sir?"

TEACHER: "Quite correct, Safa. Remember that I told you earlier that alternative definitions for a concept may exist that can lead to simpler proofs of the other properties. This is an important aspect that is usually carefully taken into account by mathematicians when they formulate definitions. We can look at this in more detail tomorrow. In any case, one more intersection is apparent, namely that between the skew kites and the cyclic quadrilaterals, let's say the *skew cyclic quadrilaterals*. Who want's to try a definition? ... Yes, Anne?"

ANNE: "The skew cyclic quadrilaterals are skew kites with opposite angles supplementary, or we can define them as cyclic quadrilaterals with at least one pair of adjacent sides equal."

TEACHER: "Very good. Which other figures that we've already had are subsets of the skew cyclic quadrilaterals? ... Yes, Jan?"

JAN: "The right kites and the trilateral trapezia and of course the squares as their intersection."

The bell rings shrilly to announce the end of the double period.

TEACHER: "Well, it unfortunately looks as if the time has caught up with us. However, we're not yet quite finished with the quadrilaterals - we will probably have to devote a couple more periods to it. For tomorrow, you must please look at the intersection between the skew trapezia and kites, and complete our classification scheme with the inclusion of the skew kites, skew trapezia, trilateral trapezia and the skew cyclic quadrilaterals, and ensure that you cover all the intersections and hierachical relationships."

Questions and Problems 1

Suggestion: Write down definitions for the quadrilaterals mentioned in the text and, if necessary, draw some sketches for clarification.

1. What are the *intersections* between the following groups of quadrilaterals? Can you prove it? (It might be useful to do Questions 1 and 2 simultaneously).

    a) Parallelograms and kites.
    b) Parallelograms and isosceles trapezia.
    c) Skew trapezia and kites.
    d) Skew trapezia and skew cyclic quads.
    e) Trilateral trapezia and right kites.
    f) Trapezia, cyclic quads and skew kites.
    g) Trapezia, cyclic quads and kites.

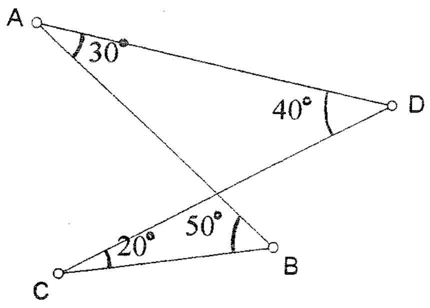
2. Redraw the classification scheme in Figure 9 to accommodate the inclusion of the skew kites, skew trapezia, trilateral trapezia and the skew cyclic quadrilaterals. (Check with your results in Question 1).

3. Carefully investigate the angle properties of a trilateral trapezium by accurate construction and measurement and/or deductive reasoning. Do you find anything interesting?

4. Carefully investigate the diagonal properties of an isosceles trapezium by accurate construction and measurement and/or deductive reasoning. What do you find?

5. Does a quadrilateral with one pair of opposite sides parallel and the other pair of opposite sides equal, but not parallel, necessarily have two pairs of adjacent angles equal? If not, can you provide a counter-example? If so, can you prove it?

6. Is the following figure a quadrilateral or not? (Motivate your answer).



7. Try and represent the classifications given in Figures 1, 3, 4, 9 and Question 2 by means of Venn-diagrams. What do you find?

Chapter 2

Defining and Classifying

The Oxford Dictionary (1974) describes the verbs "*define*" and "*classify*" respectively as "*state precisely the meaning of words or concepts*" and "*arrange in classes or groups.*" In A Dictionary of Mathematics by McDowell (1961) a "*definition*" is described as "*the explanation of the meaning of a word or phrase. A short description of something by pointing out its properties.*"

A common, but false conception is that there is only one (correct) definition for each defined object in Mathematics; in fact, several different (correct) definitions may exist. A further misconception is that mathematics always starts with definitions - indeed no definitions of mathematical objects are given *a priori* in Nature. Nor do definitions exist independently of human experience in some "*ideal*" Platonistic world, so that all we can do is to "*discover*" them. To the contrary, they are not discoveries, but human "*inventions*" for the main purpose of accurate mathematical communication. We can say that a definition is a mutual agreement amongst interested parties about what a specific object really is or what is meant by certain terminology.

The construction of definitions (defining) is a mathematical activity of no less importance than to make deductions from given definitions. In mathematics we can distinguish between two different types of defining of concepts, namely, **descriptive** (*a posteriori)* and **constructive** (*a priori*) defining (e.g. compare Krygowska, 1971; Freudenthal, 1973:461; Human, 1978:164-165; De Villiers, 1986).

Descriptive defining

"*... the describing definition ... outlines a known object by singling out a few characteristic properties*". - Hans Freudenthal (1973 : 458)

With the descriptive (*a posteriori*) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards. *A posteriori* defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems. This process is briefly summarised in Figure 12.

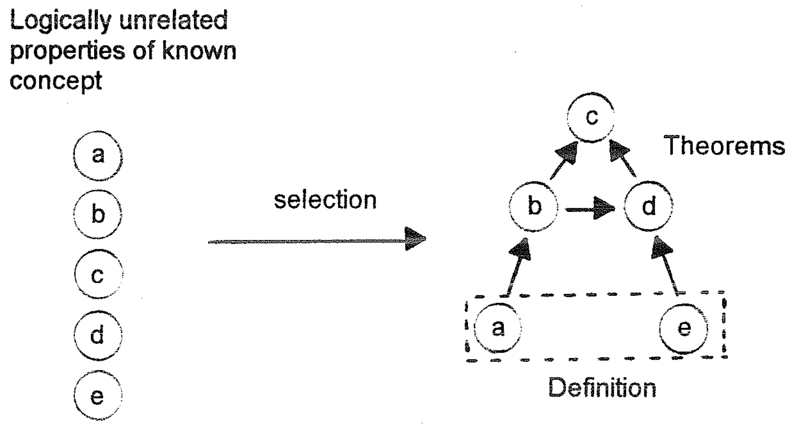


Figure 12

Let's for example consider the *a posteriori* defining of a kite, assuming that we have already discovered and formulated the following properties of it:

(a) two pairs of equal adjacent sides (or two isosceles triangles with a common base)
(b) perpendicular diagonals
(c) one diagonal is bisected by the other
(d) one pair of opposite angles is bisected by a diagonal
(e) one diagonal (the one in (d)) is a line of symmetry
(f) one pair of equal opposite angles

Before proceeding any further, first formulate one or two of your own definitions for a kite. (Check your definitions by construction and/or proof).

To define a kite we could of course list all the above properties, but it is common practice to keep a definition as short as possible. In other words, a good definition avoids unnecessary information, i.e. it must be *economical.* We therefore usually do not include in the definition *all* the properties of a set of objects being defined, but only **necessary** properties to ensure that we obtain the elements of that set. However, more serious than including too much information, is to include too little, in which case there are objects which comply with the definition, but are not elements of the set one wants to define. In other words, for a definition to be correct, it must contain **sufficient** properties to ensure that not only do we obtain the elements of the set we want to define, but only those elements (and not any others).

Accurately **construct** quadrilaterals which comply to each of the conditions in the definitions for kites in (a), (b) and (c) below. (Use ruler, compass and protracter, or if available, computer programmes such as *Sketchpad* or *Cabri-geometre*).

(a) A kite is a quadrilateral with perpendicular diagonals.

(b) A kite is a quadrilateral with two pairs of equal adjacent sides and one pair of opposite angles equal.

(c) A kite is a quadrilateral with perpendicular diagonals with one being bisected by the other.

Now carefully consider the following questions:

(i) Is the given information *sufficient* for the construction of a kite?

(ii) Did you use *all* the information given in the definition to make your construction? If not, which information did you use? Which information did you not use and why not?

The first one is not a correct definition of a kite, since it does not contain sufficient properties. For example, we can construct a diagonal and another one perpendicular on it, and then connecting the endpoints to obtain a quadrilateral as shown in Figure 13 which is clearly not a kite. Note that the statement "*a kite is a quadrilateral with perpendicular diagonals*" is a *correct statement* about a property of kites BUT that it contains too little information to be used as a **definition.** We would therefore say that "*perpendicular diagonals*" is a necessary, but not sufficient condition for kites.

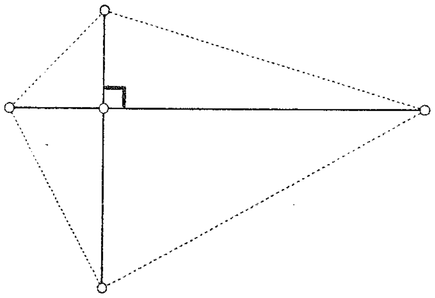


Figure 13

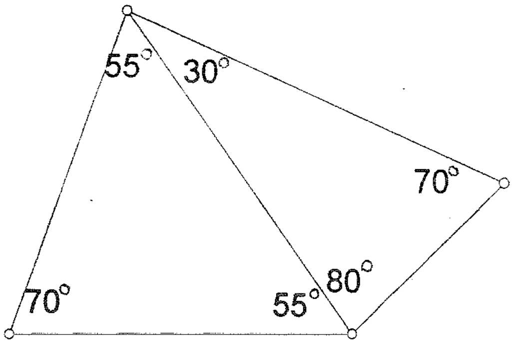


Figure 14

The second one is correct, but uneconomical as it contains too much information. In other words, the conditions are sufficient, but not all of them are necessary. But which can be left out? If we leave out "*two pairs of equal adjacent sides*" we obtain an incorrect definition, since it is possible to construct a quadrilateral with one pair of equal opposite angles which is not a kite (see Figure 14). On the other hand, we can construct a kite according to the condition "*two pairs of equal adjacent sides*" by placing a compass first at A and then at C and drawing circular arcs as shown in Figure 15. In addition, we can easily show that this condition logically implies that "*one pair of opposite angles are equal.*" For example, triangles ACB and ACD are clearly congruent (s,s,s) and therefore $\angle B = \angle D$.

The third one is a correct, economical definition as no superfluous information is supplied. In other words, we cannot leave out any of the properties, and we can construct a kite from the given conditions by first constructing two perpendicular diagonals, then placing a compass at O to chop off equal distances on the one diagonal and connecting the endpoints as shown in Figure 16. In addition, we can show that the other properties of a kite can be logically deduced from the given conditions. For example, in Figure 16 triangles AOB and AOD are clearly congruent $(s, \angle, s)$ and therefore AB=AD. In the same way it can be shown that CB=CD.

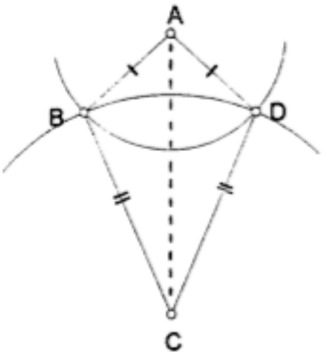


Figure 15

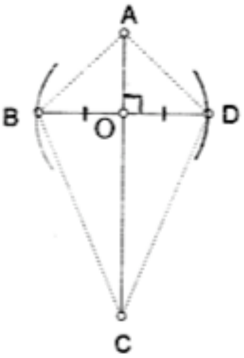


Figure 16

Definitions may also be deficient in other respects than the two aspects mentioned earlier. Critically evaluate each of the definitions for a kite below (that is, say where the mistake lies):

(a) The kites are the set of quadrilaterals with equal diagonals.
(b) The kites are the set of quadrilaterals that are shaped like a kite.
(c) The kites are the set of quadrilaterals with adjacent sides equal.

The first one contains an *incorrect property* since kites in general do not have equal diagonals. The second one is *circular*; it is unacceptable to define an object in terms of itself as it does not explain what that object is. The last one is *ambiguous* (confusing) since a quadrilateral has four pairs of adjacent sides and if *all* pairs of adjacent sides are equal we would have all sides equal, i.e. a rhombus.

Different definitions of the same concept often differ with respect to the proofs required to deduce the other properties not contained in the definition. For example, one definition of a concept may be considered more *economical* than another if the former gives rise to simpler proofs of the other properties. Compare the following three definitions for kites by deducing the other properties from it:

(a) A kite is a quadrilateral with two pairs of adjacent sides equal.
(b) A kite is a quadrilateral with one pair of opposite angles bisected by a diagonal.
(c) A kite is a quadrilateral with one diagonal a line of symmetry.

In the first one we can as before prove triangles ABC and ADC congruent (s, s, s); therefore $\angle B = \angle D$ and diagonal AC bisects the opposite angles A and C (Figure 17a). If we connect B to D then we can show triangles ABO and ADO congruent $(s, \angle, s)$, from which follows that $\angle AOB = \angle AOD = 90°$ and BO=DO; therefore the diagonals are perpendicular and the one diagonal BD is bisected. From the definition of line symmetry, it therefore now also follows that AC is a line of symmetry.

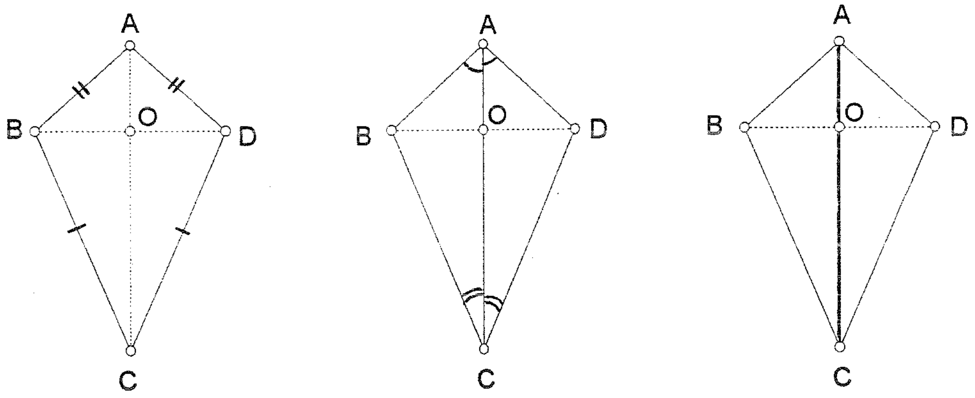


Figure 17

In the second one we similarly have to start out by proving triangles ABC and ADC congruent $(\angle, \angle, s)$, and then proceeding in the same manner as the first, the other properties can be deduced (Figure 17b). In other words, there is no real gain in terms of deductive economy in comparison to the first.

However, the third definition is deductively much more economical. For example, from the definition of line symmetry all the other properties *immediately* follow, i.e. AB=AD, BC=DC, BD $\perp$ AC, BO=DO and AC bisects angles A and C (Figure 17c). It therefore speaks for itself why we would generally prefer the latter definition.

Constructive defining

"... the algorithmically constructive and creative definition ... models new objects out of familiar ones" - Hans Freudenthal (1973 : 458).

Constructive *(a priori)* defining takes places when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process. In other words, a new concept is defined "*into being*", the further properties of which can then be experimentally or logically explored. This process is briefly illustrated in Figure 18. Whereas the main purpose or function of *a posteriori* defining is that of the ***systematization*** of existing knowledge, the main function of *a priori* defining is therefore clearly the production of ***new*** knowledge.

Let's again consider a kite and the definition for it as a quadrilateral with a line of symmetry through one pair of opposite angles. By the addition of the further constraint that it must have **two** such lines of symmetry, we clearly obtain a rhombus. Similarly, if we add to the definition of a kite the constraint that it must also be cyclic we obtain a *right kite* (see Figure 19). Furthermore, as discussed in the previous chapter we can easily show that it must then have at least one pair of opposite right angles.

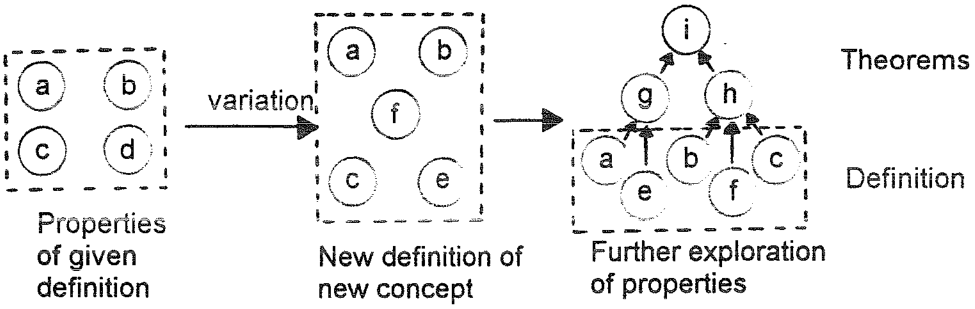


Figure 18

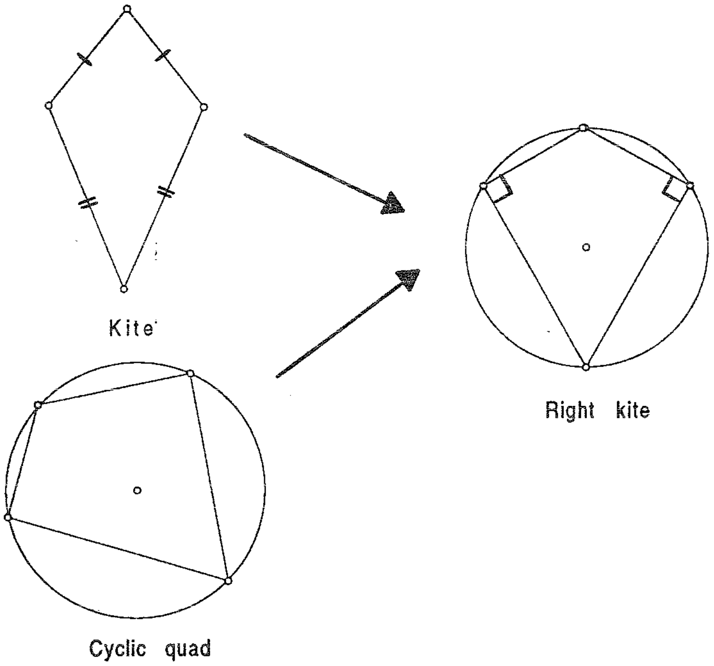


Figure 19

By retaining in turn only one property (condition) of a kite while relaxing all the other conditions, we can *a priori* define four different generalizations of a kite as shown in Figure 20. (The formulation of the definitions for *skew kites, perpendicular, bisecting* and *angle quads* are self-evident from the figure and is left to the reader). We can now explore and hopefully prove other properties of these newly defined figures (see *Questions and Problems* 2 later on).

Another way of generalization is to consider the generalization of kites to hexagons, octagons, decagons; i.e. to $2n$-gons in general with $n \geq 2$. In Figure 21 two possible generalizations to *adjacent 2n-gons* and *kite 2n-gons* are illustrated. In the former we have *n* pairs of adjacent sides equal and in the latter the property of a diagonal symmetry is maintained. As before, we can now explore and hopefully prove other properties of these newly defined figures (see *Questions and Problems 2* later on).

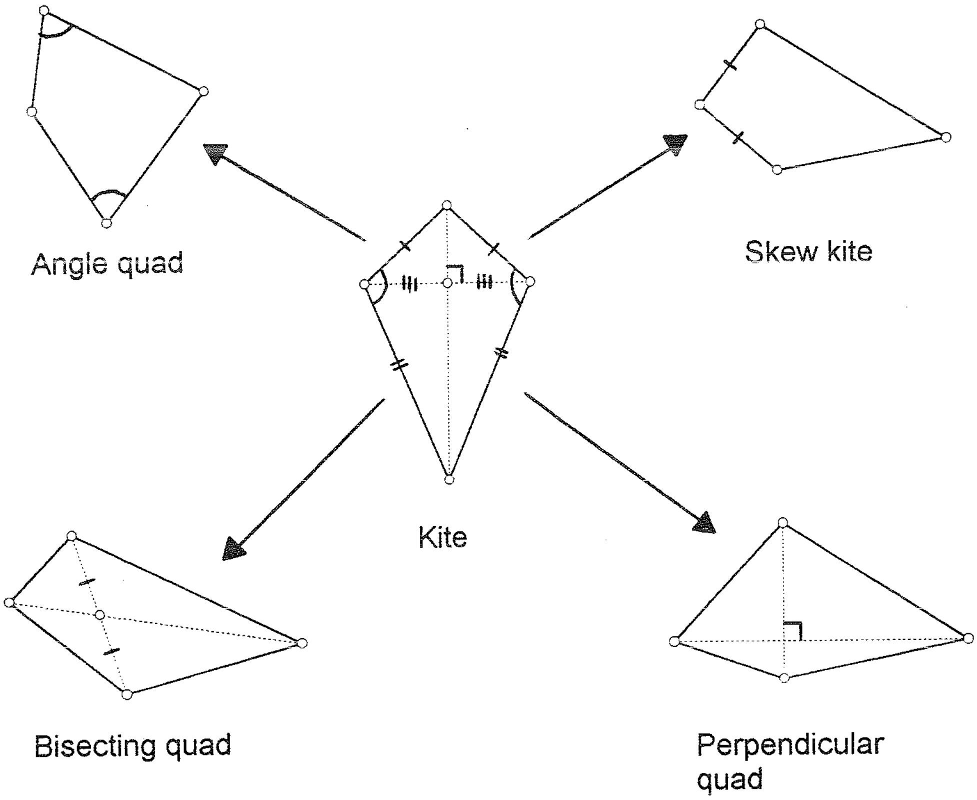


Figure 20

The relationship between classifying and defining

The classification of any set of concepts does not take place independent of the process of defining, but implicitly (or explicitly) involves defining the concepts involved, and vice versa, defining concepts in a certain way automatically also involves their classification. For example, defining a trapezium as "*a quadrilateral with **at least** one pair of opposite sides parallel*" would then imply that a parallelogram is classified as a special trapezium. On the other hand, if a trapezium is defined as "*a quadrilateral with **one** one pair of opposite sides parallel*" the parallelograms are clearly excluded from the trapezia.

We will call a definition like the first a **hierarchical** definition since it allows the inclusion of more particular concepts as subsets of the more general concept being defined. The latter we will call a **partition** definition as the concepts involved are considered to be disjoint from one another. A common misconception is that a partition definition (and classification) is mathematically "*wrong*" simply because it is partitional. However, this is not the case, since a partition definition would be quite correct, provided of course it contains sufficient information to ensure that all non-examples are excluded. For example, the following partition definition of a parallelogram which excludes rectangles, rhombi and squares may be unconventional, but it is definitely not wrong:

"A parallelogram is a quadrilateral with opposite sides parallel, but not all angles or sides equal".

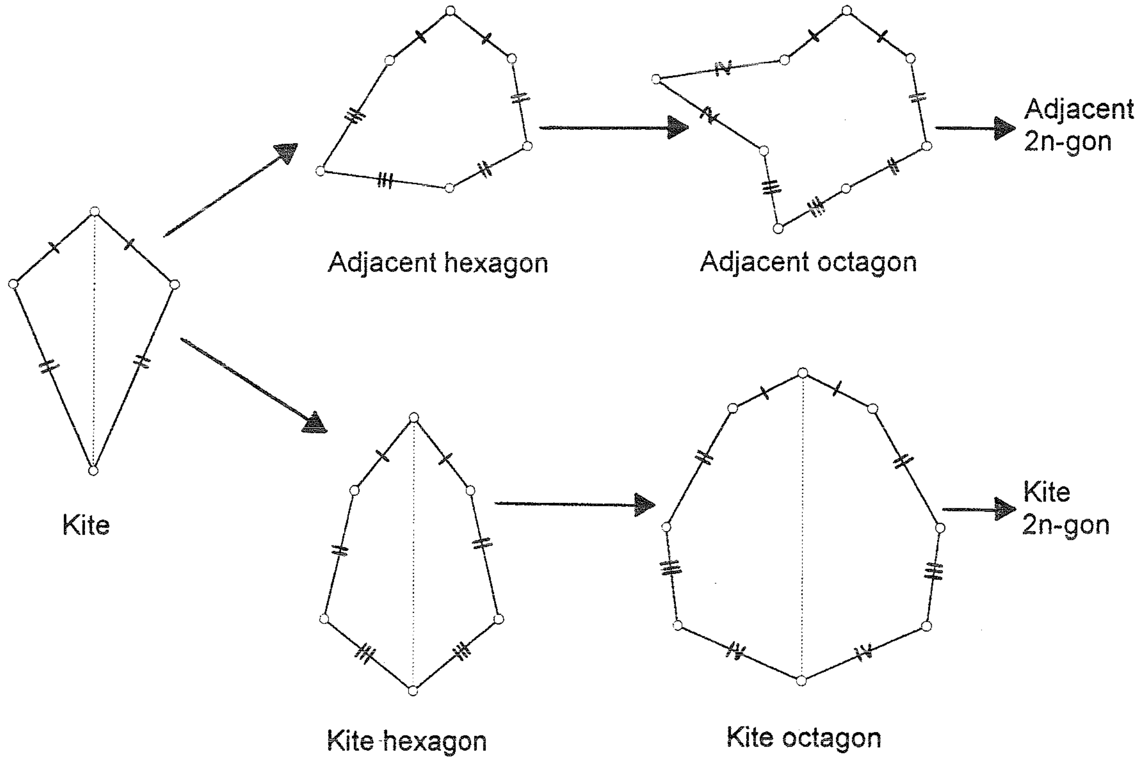


Figure 21

In fact, it is a *correct economical* (partition) definition as it contains only necessary and sufficient properties. An example of a correct, uneconomical partition definition is the following:

"A parallelogram is a quadrilateral with opposite sides equal and parallel, opposite angles equal, diagonals of different length bisecting each other, but not perpendicularly".

This partition definition clearly contains superflous (redundant) information and could be made economical in a number of different ways. An incorrect partition definition on the other hand contains insufficient or incorrect information like the following:

"A parallelogram is a quadrilateral with one pair of opposite sides parallel, but not all angles or sides equal".

"A parallelogram is a cyclic quadrilateral with one pair of opposite sides equal, but not all angles or sides equal".

Sometimes a partition classification and its corresponding definitions are necessary to clearly distinguish between concepts. For example, consider the partition classification of convex, concave and crossed quadrilaterals shown in Figure 22 with the following possible definitions for the concepts involved.

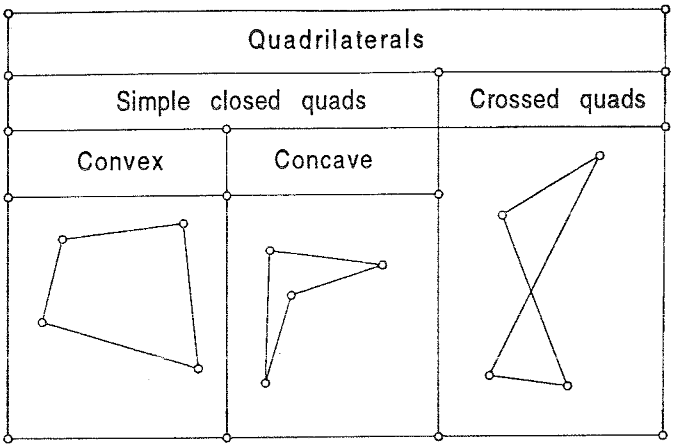
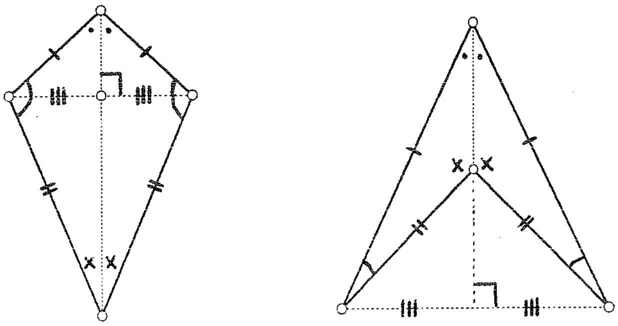


Figure 22

"A quadrilateral is any closed four sided figure in the plane with four vertices".
"A simple closed quadrilateral is a quadrilateral with sides only meeting at the vertices".
"A crossed quadrilateral is a quadrilateral with two of the sides crossing each other at a point other than the vertices".
"A convex quadrilateral is a simple closed quadrilateral with none of its angles reflexive".
"A concave quadrilateral is a simple closed quadrilateral with one of its angles reflexive".

Similarly, it is useful and necessary to partition the kites into convex and concave ones (see Figure 23). It is left as an exercise to verify that a concave kite has all the properties of a convex kite, except that one of its angles is reflexive and one diagonal falls outside the figure.



Convex

Concave

Figure 23

Partitioning is in fact a generally accepted mathematical method in many areas of mathematics, but particularly in the study of topological surfaces and spaces where the fundamental problem is the subdivision of these surfaces and spaces into different disjoint types (e.g. compare

Patterson, 1956). Since a classification and its corresponding definitions are (to a certain degree) arbitrary and not absolute, it is important to realize that the choice between a hierarchical and a partition classification is therefore often a matter of personal choice and convenience.

The fundamental question is therefore: why do we (conventionally prefer a hierarchical classification of the various quadrilaterals rather than a partition classification? Or phrased differently, what advantages does a hierarchical classification in this instance have over a partition classification?

The role and function of a hierarchical classification

Before going any further you are now requested to first complete the following exercise:

(a) Formulate a hierarchical and a partition definition for an isosceles trapezium and critically compare them. Which one is shorter, more economical?

(b) Reformulate the following well-known result from a partition perspective:
"If the midpoints E,F,G and H of the adjacent sides of any quadrilateral are consecutively connected, then EFGH is a parallelogram."

(c)(i) If we classify a rectangle as a parallelogram, is it necessary to prove that the diagonals of a rectangle bisect each other?
(ii) If we partition the rectangles from the parallelograms is it necessary to prove that the diagonals of a rectangle bisect each other?

Some of the most important functions of a hierarchical classification as indicated above are:

* it leads to more economical definitions of concepts and formulation of theorems
* it simplifies the deductive systematization and derivation of the properties of more special concepts
* it often provides a useful conceptual schema during problem solving
* it sometimes suggests alternative definitions and new propositions
* it provides a useful global perspective

Each of these will now be discussed in slightly more detail.

Economical definitions and formulations of theorems

Economy of definition and of formulation of theorems is probably one of the most important advantages of a hierarchical classification. For example, a conventional hierarchical definition for a paralellogram is shorter than the partition definition provided earlier in this section, since the latter has to include additional properties to exclude the rectangle, rhombi and squares. Also consider the defining of an isosceles trapezium. A hierarchical definition which includes the rectangles, trilateral trapezia and squares would be to say that it is any quadrilateral with an axis of symmetry through (at least) one pair of opposite sides. A partition definition on the other hand, which excludes the rectangles, trilateral trapezia and squares, would have to include the additional conditions that it may not have a right angle nor three sides equal.

A partition classification often also makes the formulation of certain theorems clumsy and cumbersome. Consider for example the following two formulations of well-known results from a partition perspective:
"If the midpoints E,F,G and H of the adjacent sides of any quadrilateral are consecutively connected, then EFGH is a parallelogram, rectangle, rhombus or square."
"The exterior angle of a cyclic quadrilateral, isosceles trapezium, right kite, trilateral trapezium, rectangle or square is equal to the opposite interior angle."

Simplification of deductive structure
By classifying (and defining) a concept A as a subset (special case) of a concept B, it becomes unneccessary to repeat any of the proofs of the properties of concept B for concept A, as they are automatically implied for A by the hierarchical inclusion. For example by classifying a rectangle as a parallelogram, all the theorems which have already been proved for parallelograms are immediately made applicable to the rectangles (and squares). In other words, it is unnecessary to prove for instance that the diagonals of a rectangle (and square) bisect each other or that their opposite sides are equal.

In contrast, if the rectangles were to be excluded from the parallelograms, one would strictly speaking have to again prove that the above properties are also true from the chosen definition for rectangles, whatever it may be (e.g. suppose "*a quadrilateral with four equal angles, but not all sides equal*"). In addition, if the squares are also excluded from the rectangles by a partition definition like the preceding one, the proofs of these properties would have to be repeated again for the squares.

Similarly by hierarchically classifying squares as rhombi and rhombi as kites, it is unnecessary to again prove for instance that the diagonals of a square or rhombus are perpendicular, since this is an easily proved property of the kites. In contrast, a partition classification would require one to repeat the proofs for squares and rhombi. Another good example is that if we hierarchically classify rectangles and squares as isosceles trapezia, it is unnecessary to prove that they have equal diagonals, as this is also an easily proved property of the isosceles trapezia. Apart from the economy of definition or formulation, a hierarchical classification therefore also results in an economical deductive system.

A useful conceptual schema during problem solving

A hierarchical class inclusion is often also useful during problem solving; in particular for proving riders. For example, suppose one wants to prove that a kite with one pair of opposite sides parallel is a rhombus. Using the fact that the rhombi are the **intersection** of the kites and parallelograms, it is **sufficient** therefore to prove that the figure is a parallelogram, since any kite with both pairs of opposite sides parallel must be a rhombus. Other examples can be found in *Questions and problems 1* and will also presented later on.

Alternative definitions and new propositions

Consideration of the hierarchical relationships between concepts may sometimes suggest alternative definitions and new propositions. If for instance concept A is the intersection for two other concepts B and C, then it must obviously possess all the properties of both concepts B and C. By now considering different combinations of the properties of concepts B and C, alternative definitions for concept A, or new propositions, may be suggested.

For example, since any isosceles trapezium is cyclic, and its diagonals are equal, the following alternative definition for isosceles trapezia is possibly suggested:
"An isosceles trapezium is a cyclic quadrilateral with equal diagonals".

Alternatively, it can be formulated as a proposition:
"If a cyclic quadrilateral has equal diagonals then it is an isosceles trapezium".

The keeping in mind of a hierarchical classification can also sometimes enable the generalization of certain results. If for example, one came across certain results involving the construction of equilateral triangles or squares, one might then consider the possible extention of those results to the construction of similar isosceles triangles or similar rectangles and rhombi. Several examples of these will be given later on.

A useful global perspective

A hierarchical classification provides a useful global perspective which may lead to a more *cohesive* perspective on the underlying relationships between concepts, and therefore also to better *retention*. In addition, it is aesthetically pleasing and insightful to see how the various intersections between more general concepts produce the properties of more special concepts.

For example, since the rhombi are the intersection between the kites and parallelograms, it immediately follows from the diagonal properties of kites and parallelograms that the diagonals of a rhombus would perpendiculary bisect each other.

Similarly, since the rectangle are the intersection between the parallelograms and isosceles trapezia, it immediately follows that a rectangle must have opposite angles equal (parallelogram property) as well as adjacent angles equal (isosceles trapezium property), from which we get the familiar property that all its angles are equal. In the same way, the rectangles "*inherit*" equal diagonals from the isosceles trapezia, as well as bisecting ones from the parallelograms.

In conclusion we therefore see that the reason why a hierarchical classification of quadrilaterals is generally preferred to a partition classification is that the former is much more useful and not because the latter is "*wrong*".

Questions and Problems 2

1. List **all** the properties of any quadrilateral, say a parallelogram. Is it possible to prove **all** the properties of a parallelogram?

2. State for each of the following conditions whether it is (verify with proofs or illustrate with counter-examples if necessary):
   (i) necessary and sufficient or
   (ii) necessary but not sufficient
   (iii) sufficient but not necessary

(Note : assume for this exercise that we're restricted to only convex and concave cases).

(a) A ________________ condition that a kite is a rhombus is that it has one pair of opposite sides parallel.

(b) A ________________ condition that a cyclic quadrilateral has at least one pair of opposite sides parallel, is that at least three of its sides must be equal.

(c) A ________________ condition that a kite is cyclic is that it has at least one pair of opposite right angles.

(d) Equal diagonals is a ________________ condition for a quadrilateral to be a rectangle.

(e) Equal diagonals is a ________________ condition for a quadrilateral to be an isosceles trapezium.

(f) One pair of equal opposite angles is a ________________ condition for a quadrilateral to be a kite.

(g) Three angles of a quadrilateral correspondingly equal to three angles of another quadrilateral, is a ________________ condition for them to be similar.

(h) One pair of opposite sides parallel and one pair of opposite angles equal is a ________________ condition for a quadrilateral to be parallelogram.

(i) One pair of opposite sides parallel and the other pair equal is a ________________ condition for a quadrilateral to be a parallelogram.

(j) One pair of opposite sides equal and one pair of opposite angles equal is a

_________________ condition for a quadrilateral to be a parallelogram.

(k) A point of half-turn symmetry is a _________________ condition for a quadrilateral to be a parallelogram.

(l) Two axes of symmetry through each of the two pairs of opposite sides is a _________________ condition for a quadrilateral to be a rectangle.

(m) Two axes of symmetry, one through a pair of opposite sides and another through a pair of opposite angles, is a _________________ condition for a quadrilateral to be a square.

(n) Two pairs of opposite angles supplementary and one pair of opposite sides parallel is a _________________ condition for a quadrilateral to be an isosceles trapezium.

(o) Perpendicular diagonals and one pair of opposite sides parallel is a _________________ condition for a quadrilateral to be a rhombus.

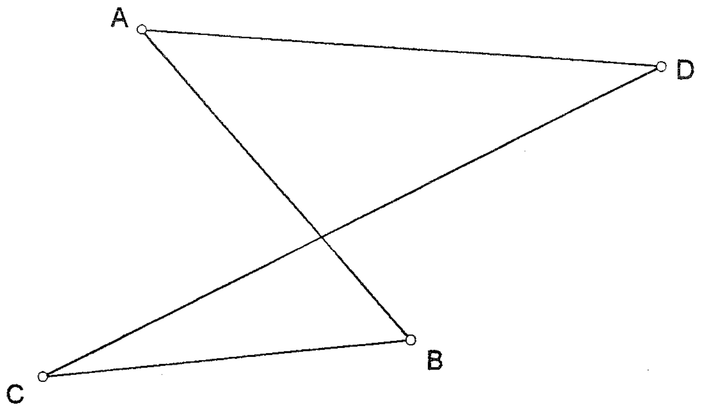
3. Critically evaluate each of the definitions below:
   (a) The squares are the set of quadrilaterals that are square.
   (b) The parallelograms are the set of quadrilaterals with all their sides parallel.
   (c) The rhombi are the set of quadrilaterals with all sides and angles equal.
   (d) The isosceles trapezia are the set of quadrilaterals with perpendicular diagonals.

4. Make a list of all the properties of an isosceles trapezium and then formulate:
   (i) an *incomplete* definition for an isosceles trapezium (one which has necessary but not sufficient properties).
   (ii) a correct, *uneconomical* definition for an isosceles trapezium.
   (iii) three correct, *economical* definitions for an isosceles trapezium.

5. Repeat the above exercise with rectangles, rhombi and squares.

6. Formulate (correct, economical) partition definitions for rectangles, rhombi, cyclic quads and bisecting quads.

7.(a) Carefully consider which angles are the *interior* angles of a crossed quadrilateral (see figure below). What is the sum of the interior angles of a crossed quadrilateral?



(b) Which are the *opposite* sides and angles of a crossed quadrilateral?

8. Try formulating alternative definitions for convex, concave and crossed quadrilaterals to those given the text.

9. Carefully consider whether we can find convex, concave or crossed examples of the following quadrilaterals (make drawings):
   (a) Bisecting quad
   (b) Perpendicular quad
   (c) Trapezium
   (d) Cyclic quad
   (e) Parallelogram
   (f) Right Kite
   (g) Rhombus
   (h) Rectangle

10. Repeat no. 2 with the additional consideration of crossed cases.

Note: In all the following exercises you have to consider convex, concave and crossed cases.

11. Are the kites the intersections between the following sets of quadrilaterals? (Check by construction and/or provide proofs. Give counter-examples where necessary).
   (a) the bisecting and perpendicular quads
   (b) the bisecting quads and skew kites
   (c) the angle quads and skew kites
   (d) the angle and the perpendicular quads
   (e) the skew and perpendicular quads
   (f) the bisecting and angle quads

12. Find a formula for the area of a convex perpendicular quad in terms of its diagonals. Is this formula also valid for the concave and crossed cases?

13. Prove that in a bisecting quad the diagonal that bisects the other also bisects its area.

14. Is a cyclic quadrilateral with equal diagonals always an isosceles trapezium? Investigate. If true, provide a proof. If not, provide a counter-example.

15. Is a trapezium with equal diagonals always an isosceles trapezium? Investigate. If true, provide a proof. If not, provide a counter-example.

16 What can you say about the sums of the two pairs of *alternate* sides of a kite $2n$-gon? (Alternate sides are those sides when moving along the perimeter of a polygon that are separated from each other by only one side; i.e. in other words, every second one).

17. Consider some ways of generalizing the concept parallelogram.

18. Is any quadrilateral ABCD an isosceles trapezium if it has opposite sides equal and the diagonals intersect in O so that AO/OC = DO/OB? Investigate. If true, provide a proof. If not, provide a counter-example.

Chapter 3

Mathematical discovery and proof

"...mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations." - Imre Lakatos (1976:5)

Is new mathematics discovered or created? How is it discovered or created? What are the heuristics behind new developments? How can one discover or invent new mathematics for oneself? Or does one have to be exceptionally brilliant? Why do we prove things in mathematics?

Although it is impossible to answer the above questions in full in one chapter, this chapter will nevertheless attempt to at least partially address some of these issues with the hope that it will not only be useful to aspiring mathematicians, but also might inspire some interest in those who are not.

Discovery or creation?

There have been many lengthy debates amongst philosophers, mathematicians and mathematics educators about whether mathematics is actually *created* or *discovered*. The former view emphasizes the human element; our freedom of choice and that it is our invention. The latter view assumes that mathematics already exists *"out there somewhere"* in an ideal world, independent of human existence (the so-called Platonistic view) and that we therefore discover mathematics. Which view is correct?

The fact of the matter is: neither! New mathematics is created, as well as discovered. When we formulate definitions, classify, choose terminology and symbols, we are of course creating: we have a certain freedom of choice; we are the inventors. For example, if we consider any specific concept like *"line symmetry"*, one could argue that it is simply a human invention to give meaning to certain observed regularities in geometrical figures.

However, one could also argue that this concept had been out there all the time simply waiting to be abstracted from the above-mentioned regularities by someone. Furthermore, can we really say that it is solely our creation or invention that there is for example a constant ratio between the circumference of a circle and its diameter, when this observation is *independent* of whatever unit of measuring length we decided to choose? The value of π is fixed for a chosen unit of measurement and we have no alternative choice but to accept that fact. For example, consider the futile attempts by legislators in the State of Indiana who in 1897 tried to fix by law the value of π at 3 with Law no. 26 (see Eves, 1953:102).

In fact, it seems reasonable to assume that if there were some alien intelligence on a far distant planet, they would arrive at the same conclusion, irrespective of their language, mathematical symbolism and chosen unit of measurement (even though the latter may be totally incomprehensible to us). Interestingly, one of the Voyager probes sent past Pluto and out of our solar system, carries a plaque inscribed with geometrical configurations representing some important theorems, which if it were to land in the *"hands"* of some alien intelligence, would hopefully be recognizable to them as mathematics.

Some important mental attitudes

The great mathematician Gauss is reputed to have once been asked to what he owed his phenomenal mathematical output. He then humbly replied something along these lines:
"If other people had spent as much time on mathematics as I have, they would have made the same discoveries."

Far more than some peculiar innate intelligence, it seems that there are at least five essential mental attitudes to making new mathematical advancements; namely, sustained attention, commitment, detachment, flexibility and confidence (compare Schmalz,1988).

Sustained attention

By attention is meant here a discipline of concentrating on just one thing - one must learn to center, to focus one's attention to the exclusion of other things. Although the daily work of a mathematician is often dull and dreary, this work is *preparatory* to intuitive insights and new discoveries. One must learn to be attentive, fully engaged in looking, pondering and concentrating on one's work.

Commitment

One has commitment to something if it matters deeply to one and satisfies one's quest for meaning. Such commitment to mathematics and a constant quest for its meaning is often reflected in the writings of great mathematicians. For example, Bertrand Russell (1967:37) reports in his autobiography:
"At the age of eleven, I began Euclid with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world - mathematics was my chief interest, and my chief source of happiness."

Detachment

Detachment can be defined as not being concerned about outcomes, but it is not the same as indifference; it does not mean the absence of energetic commitment. It requires a certain aloofness or distancing from one's work; one should be able to carefully assess one's own work objectively, be open to possible criticism and counter-examples, and on the look-out for preconceived judgements that could distort one's focus.

Flexibility

The willingness to try out new ideas and to explore new avenues of thought is probably one of the most important mental attitudes to have: to ask *"what if?"* questions, to conjecture and test, and then to reformulate and critically reassess in the light of counter-examples. For example, faced with any result one could always ask if it can be generalized, specialized or extended to other areas by analogy or other means.

Confidence

Closely associated with a willingness to experiment is a growth in inner confidence which allows an inner freedom enabling one to leave a traditional path or attempt a new generalization, method or procedure. Someone who is insecure, who lacks confidence in him/herself, may be unwilling to make conjectures, to ask *"what if?"* questions, because of the risk of making mistakes. It therefore requires a certain willingness to make and accept the occurrence of mistakes in one's quest for new knowledge.

The logic of mathematical discovery and proof

Logically, mathematics is based upon the following fundamental axiom: *"Something is true (T), if and only if, it can be (deductively) proved (P)"* . It is useful to represent it in the following equivalent logical forms:

(a) the forward implication ($T \Rightarrow P$): if something is true, then it can be proved.
(b) the converse ($P \Rightarrow T$): if something has been proved, then it is true.
(c) the inverse ($T' \Rightarrow P'$): if something is false, then it cannot be proved.
(d) the contrapositive ($P' \Rightarrow T'$): if something cannot be proved, then it is false.

Unfortunately in textbooks and teaching only the converse is usually conveyed; in other words, that we must first prove results, before we can accept them as true. However, in actual mathematical research the forward implication, its inverse and contrapositive often play a far greater role in motivating and guiding our actions. For example, suppose we were to make a conjecture and then test it by some cases. If the conjecture is not supported by these cases, we reject it as false and according to the inverse do not even bother trying to prove it. On the other hand, if it is supported by these cases, we might start believing it to be true, which according to the forward implication then gives us the encouragement to start looking for a proof. However, if after a while we are not successful in producing a proof, we might start doubting the validity of the conjecture according to the contrapositive, and then consider some more cases, after which the whole process is of course repeated.

The role and function of quasi-empirical methods

With *"quasi-empirical"* methods is essentially meant here all non-deductive methods which are related to an empirical or experimental approach and includes intuition and induction. For example, such methods typically occur when:

- conjectures or theorems are ***evaluated*** by special cases, numerical substitution, accurate construction and measurement, folding and cutting, graphing, etc.
- conjectures, generalizations or conclusions are *formulated* on the basis of intuition and/or experience like the aforegoing.

The following two main functions of such methods can be distinguished although they are inter-related:

- obtaining certainty
- provision of counter-examples

Obtaining certainty

(Suggestion: The reader is strongly advised to construct and investigate the properties of Figures 24 and 25 with dynamic software like *Cabri Geometre* or *Geometer's Sketchpad*).

Consider the first kite in Figure 24. Connect the midpoints of the sides to obtain a quadrilateral EFGH. What do you notice?

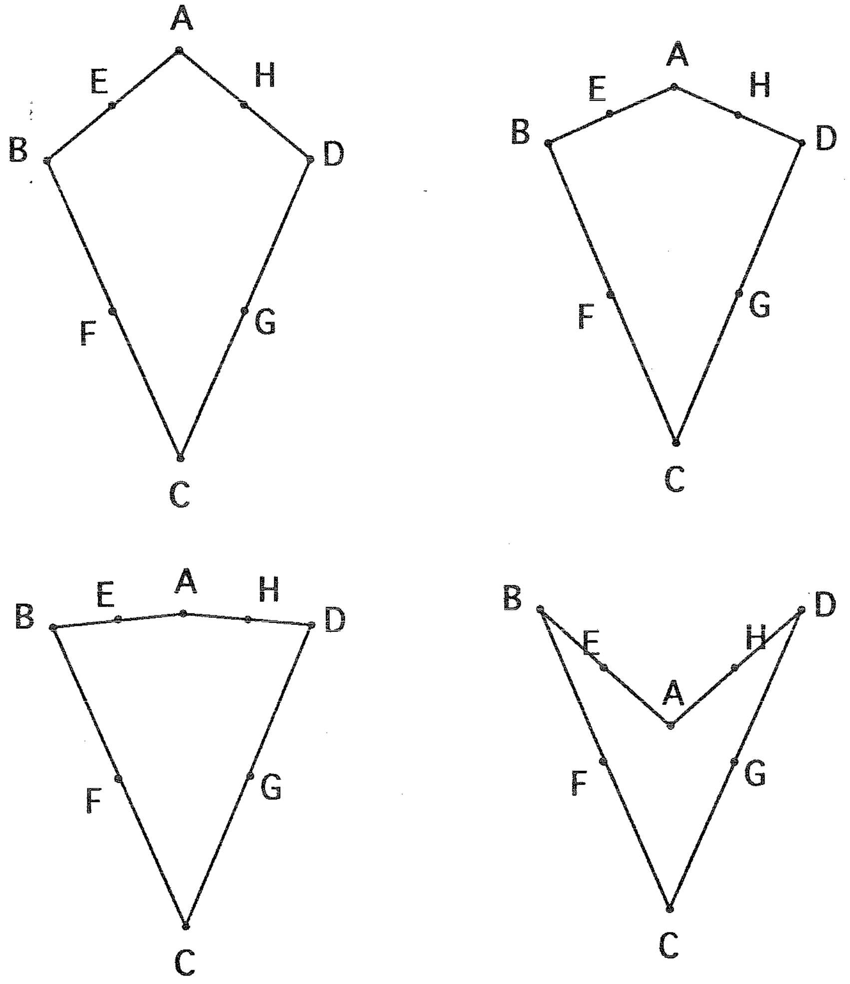


Figure 24

No doubt you noticed that EFGH appears to be a rectangle. Perhaps you even measured an angle or two to make sure. Did you expect it to be a rectangle or were you surprised? Do you think it will always be true? Or do you think it was just coincidence? Do you perhaps first want to test some more? If so, also connect E,F,G and H in the other three figures in Figure 24. Does this confirm or refute your present suspicion?

You are now asked to be truly honest with yourself and to ask yourself how **certain** you are that this result would be true in any kite. Do you largely doubt its truth or are you reasonably certain? Can you give a percentage to it? 50%, 70%, 90%, 99%, 100%? How would you become more certain? Would you be more certain if you could construct a couple more on your own?

Let's look at another example. Consider Figure 25 which shows right triangles ABC with equilateral triangles DAB, EBC and FCA constructed on the sides. Draw DC, EA and FB in the first figure. What do you notice?

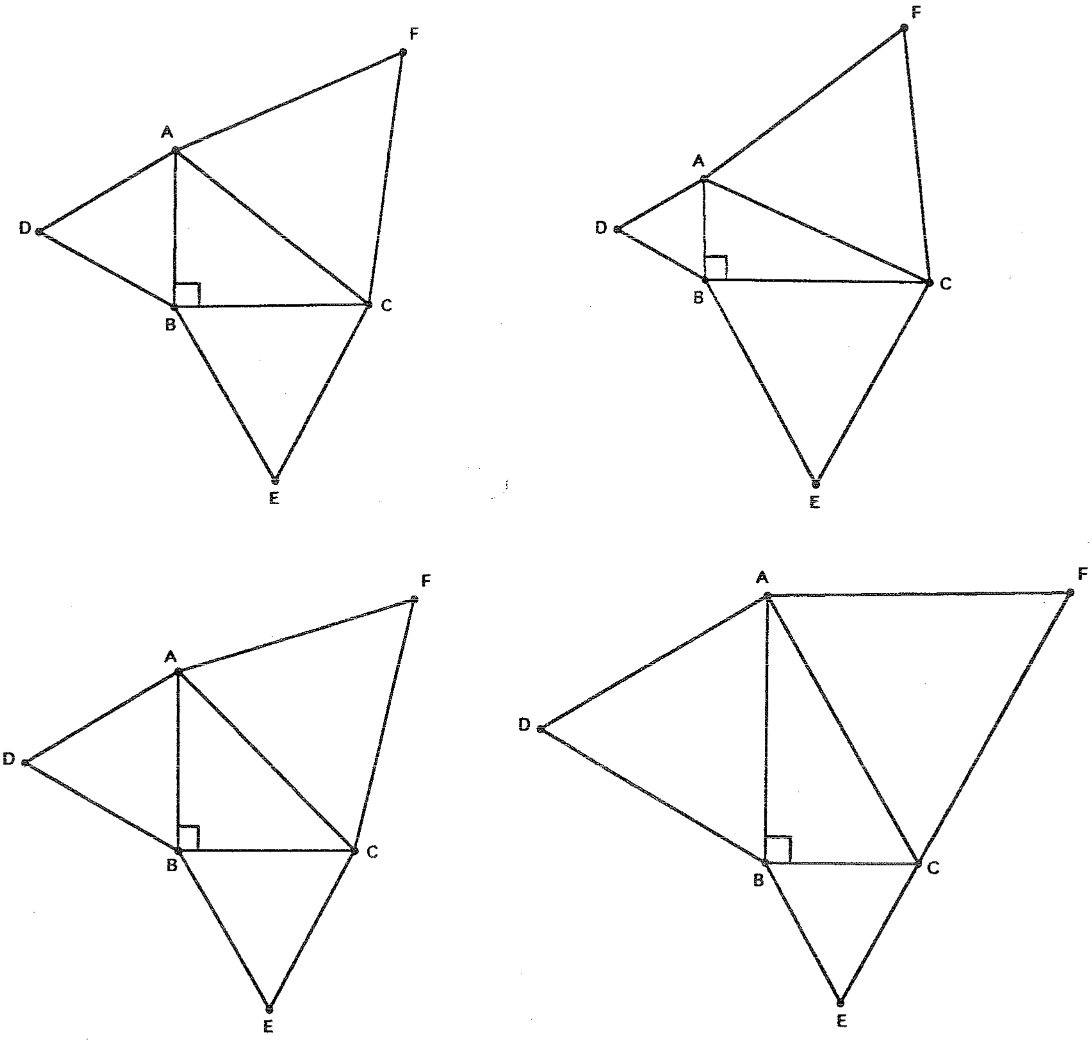


Figure 25

No doubt you noticed that DC, FB and EA are concurrent. Repeat the previous set of questions for yourself and honestly reflect on your personal certainty or uncertainty, and your need for further conviction.

The majority of readers are probably now reasonably certain about the general truth of these two results, even though no deductive reasoning was involved at all. In fact, a very high level of confidence can sometimes be obtained by only using empirical, inductive, analogical, intuitive or heuristic means. Furthermore, it is often precisely such *a priori* conviction that motivates us to start looking for a proof, and not so much because we doubt that a specific result is true. (*"If it is true, then we should be able to prove it"*).

An important recent development is the arrival of powerful new computer software like *Cabri Geomètre* and *Geometer's Sketchpad* with which one can dynamically produce many examples by simply clicking on one of the points of a geometric configuration and dragging it across the screen. In fact, *Cabri* has a sophisticated *checking facility* that can check whether certain properties like concurrency, perpendicularity, parallelism and collinearity are *true in general.* If not, it produces a counter-example. Using this facility, it is easy to confirm the above two results to be true in general.

That this kind of quasi-empirical conviction often precedes and motivates a proof is borne out in the history of mathematics, i.e. by the frequent heuristic precedence of results over arguments, of theorems over proofs. For example, Gauss is reputed to have complained: *"I have had my results for a long time, but I do not yet know how I am to (deductively) arrive at them"* . Two famous 20th century mathematicians underscore this idea as follows:

"The mathematician at work ... arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof." - Paul Halmos (1984:23)

"...having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong confidence in the theorem. Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it." - George Polya (1954:83-84)

The practice of first evaluating an unknown conjecture by the consideration of specific cases is probably about as old as mathematics itself, and is still actively utilized in modern day research. Neubrand (1989:4) for example writes as follows about the recent proof of Bieberbach's conjecture (1916) (now De Branges' theorem (1984)):
"As in many other cases, in this example mathematicians first started with the consideration of special cases, restricted cases, etc., in order to convince themselves of the possibility of the validity of the conjecture."

Provision of counter-examples

It is important to realize that counter-examples are mainly produced by quasi-empirical testing, and usually not by deductive reasoning. Consider for example the following statement from the previous chapter, namely: *"a quadrilateral with equal diagonals is an isosceles trapezium"*. To construct a counter-example for this statement it is necessary to check quasi-empirically whether sufficient information is provided for the production of an isosceles trapezium. If we construct two equal diagonals and let them intersect in an arbitrary manner as shown in Figure 26, we easily find that the constructed figure is not necessarily an isosceles trapezium. Similarly, we would not use deduction to construct a counter-example for the following statement: *"a quadrilateral with perpendicular, bisecting diagonals is a square"*, but quasi-empirical testing.

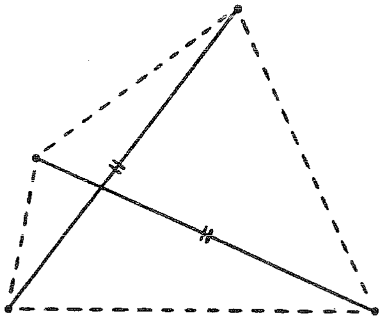


Figure 26

When we are therefore dealing with a totally unknown conjecture, first testing it quasi-empirically serves one of the following purposes:

(1) the immediate construction of a counter-example if it is false, or

(2) the attainment of a reasonable amount of certainty (conviction) which then encourages us to start looking for a proof.

In real mathematical research counter-examples are sometimes even discovered for a proved theorem which then necessitates the reformulation of the theorem and/or its proof. For example, in *Questions and Problems 1, no.6* a crossed quadrilateral was given as a counter-example to the well-known theorem proved at school: *"the sum of the angles of any quadrilateral is equal to* 360°".

Usually one's first reaction to such a counter-example is one of *"monster-barring"* in support of the theorem, i.e. to reject a crossed quadrilateral as a quadrilateral. One would therefore try to define a quadrilateral in such a way that crossed quadrilaterals are excluded. Lakatos (1976:16) eloquently describes a similar situation after the discovery of a counter-example to the Euler-Descartes theorem for polyhedra by the characters in his book:

"DELTA: But why accept the counter-example? We proved our conjecture - now it is a

theorem. I admit that it clashes with this so-called 'counter-example'. One of them has to give way. But why should the theorem give way, when it has been proved? It is the 'criticism' that should retreat. It is fake criticism. This pair of nested cubes is not a polyhedron at all. It is a ***monster****, a pathological case, not a counter-example.*
GAMMA: Why not? A ***polyhedron is a solid whose surface consists of polygonal faces****. And my counter-example is a solid bounded by polygonal faces.*
DELTA: Your definition is incorrect. A polyhedron must be a ***surface****: it has faces, edges, vertices, it can be deformed, stretched out on a blackboard, and has nothing to do with the concept of 'solid'.* ***A polyhedron is a surface consisting of a system of polygons."***

From the above brief extract, we also see that refutation by counter-example in such cases depends on the meaning of the terms involved and consequently definitions are frequently proposed and argued about. However, since we decided to consider a crossed quadrilateral as a quadrilateral, we therefore simply have to restrict the angle-sum theorem for quadrilaterals to only **simple closed** quadrilaterals. (Also note that the usual proof assumes that at least one of the diagonals falls inside the quadrilateral).

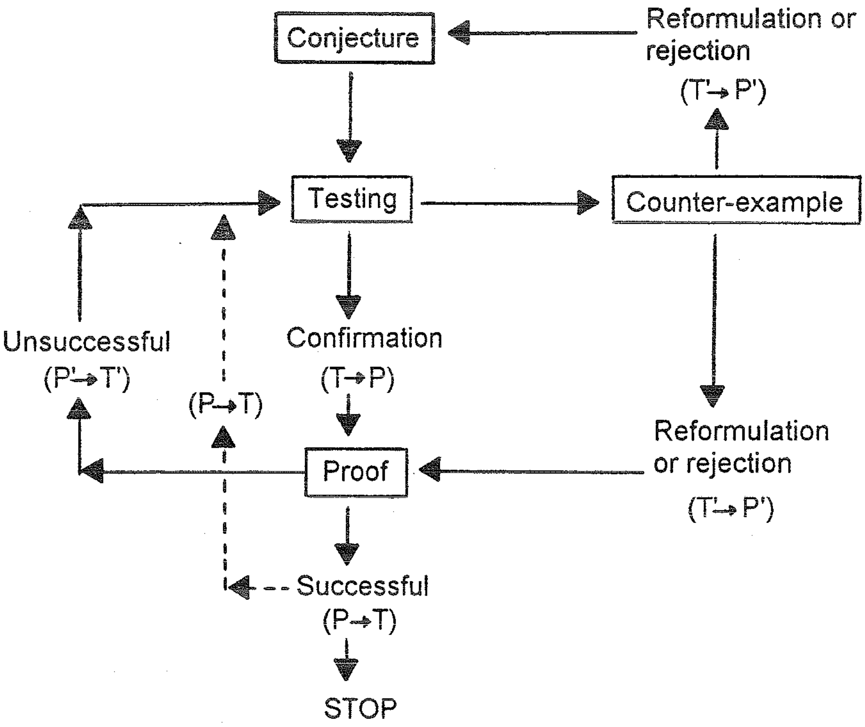


Figure 27

This process of proofs, refutations and reformulations can sometimes go through several cycles as represented in Figure 27. Two famous historical examples which spanned many decades are the Euler-Descartes theorem and Cauchy's theorem about the continuity of the limit of any convergent series of continuous functions.

Quasi-empirical testing is also useful for identifying incorrect assumptions in otherwise completely valid reasoning. Recall for example the *"proof"* given in *Solutions 2, no.2(j)* that a quadrilateral with one pair of opposite sides equal and one pair of opposite angles equal is a parallelogram. Although no fault can be found with the argument itself, it is based on the false assumption that the constructed perpendiculars always fall inside the quadrilateral, and which only becomes apparent through actual construction and measurement. Many ingenious paradoxes can arise by virtue of construction errors or mistaken assumptions in diagrams. Consider for example the following paradox.

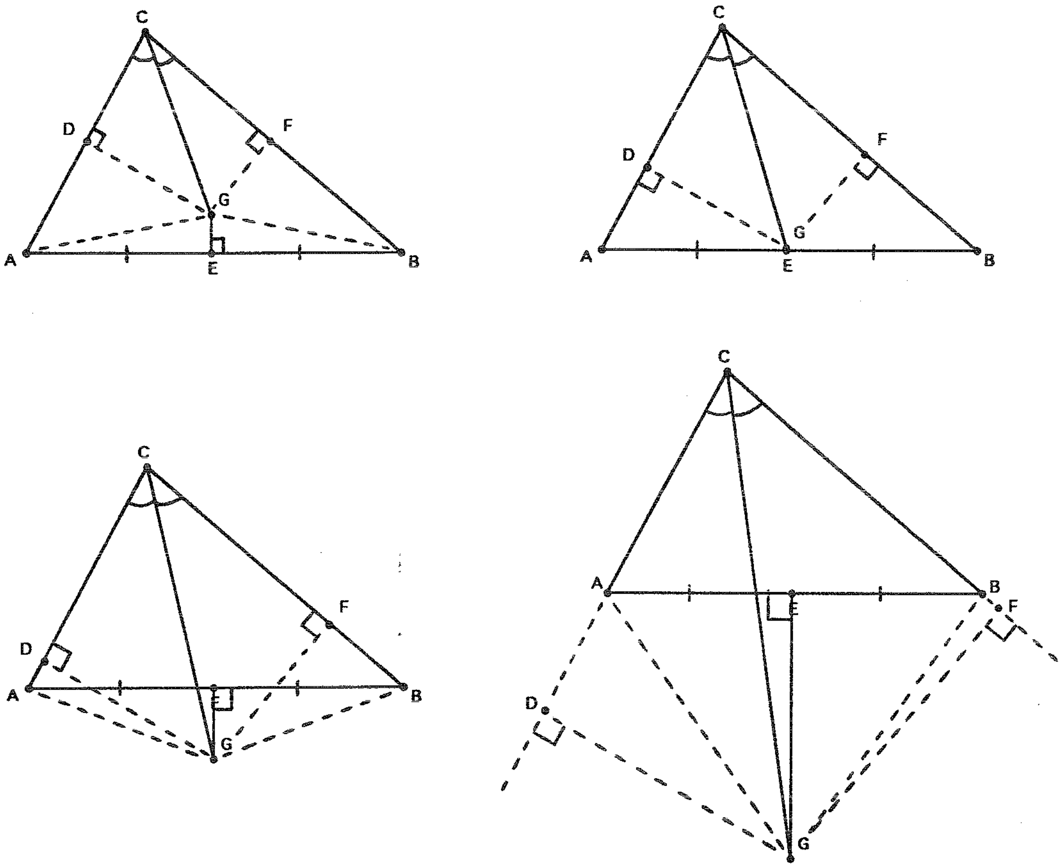


Figure 28

Every triangle is isosceles.

Take an arbitrary triangle ABC and construct the bisector of angle C and the perpendicular bisector of AB (see Figure 28). From G their point of intersection, drop perpendiculars GD and GF to AC and BC respectively and draw AG and BG. We now distinguish between four possible cases regarding the positions of G and the perpendiculars GD and GF. Consider the first figure. Triangles CGD and CGF are congruent $(\angle,\angle,s)$ and therefore DG = GF. This implies that triangles GDA and GFB are congruent $(90^\circ,s,s)$ and therefore DA = FB. But from the first congruency we also have CD = CF and by addition we have CD + DA = CA = CF + FB = CB. Thus triangle ABC is isosceles. It is now left to the reader to verify that the same conclusion can be drawn from the other cases.

What is the problem? Where is the mistake? The problem lies with the inaccuracy of the drawings. Had we actually at the start constructed, by means of computer, or ruler and compasses, the angle bisector, the perpendicular bisector and the two perpendiculars, we would have found as shown in Figure 29 that *one* of the points D and F always falls *inside the triangle and the other outside*, which completely invalidates the above *"proofs"*! This episode shows how easily a logical argument can be swayed by what the eye sees in a figure and so emphasizes the importance of quasi-empirical testing (i.e. the accurate construction of some examples), noting with care the relative positions of points essential to the proof.

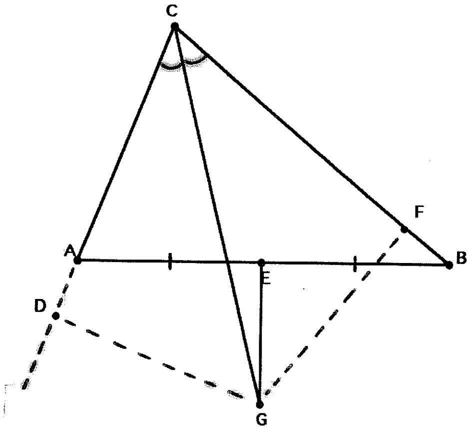


Figure 29

Proof as a means of explanation and discovery

"A good proof is one that makes us wiser." -Yu Manin (1981:107)

Traditionally proof in the classroom and lecture theatre is solely presented as a means of obtaining certainty/conviction. Although not denying the usefulness of proof in obtaining additional certainty, mathematicians often construct proofs for quite other reasons than simply that of verification/conviction. One of these is to try and explain or understand **why** some results are true.

Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of quasi-empirical verification (e.g. accurate construction and measurement by hand or computer, numerical substitution, etc.) this generally provides no satisfactory explanation why that conjecture may be true. It merely confirms that it is true, and even though the consideration of more and more examples (or the use of *Cabri's* property checker) may increase one's confidence still more, it gives no psychological satisfactory sense of *illumination*, i.e. an insight or understanding into how and why it is the consequence of other familiar results. For example, in their book **The Mathematical Experience** Davis & Hersh present very convincing heuristic evidence in support of the still unproved Riemann Hypothesis but then express a burning need for explanation as follows:

"It is interesting to ask, in a context such as this, why we still feel the need for a proof... It

seems clear that we want a proof because ... if something is true and we cannot deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding ***why*** *the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it* ***is true.***" - Davis & Hersh (1983:368)

Let's now consider the conjecture shown in Figure 24. Before proceeding further, first try and deductively **explain** it for yourself. Look carefully at your proof. Which properties of a kite did you use and which did you not? What can you conclude from this?

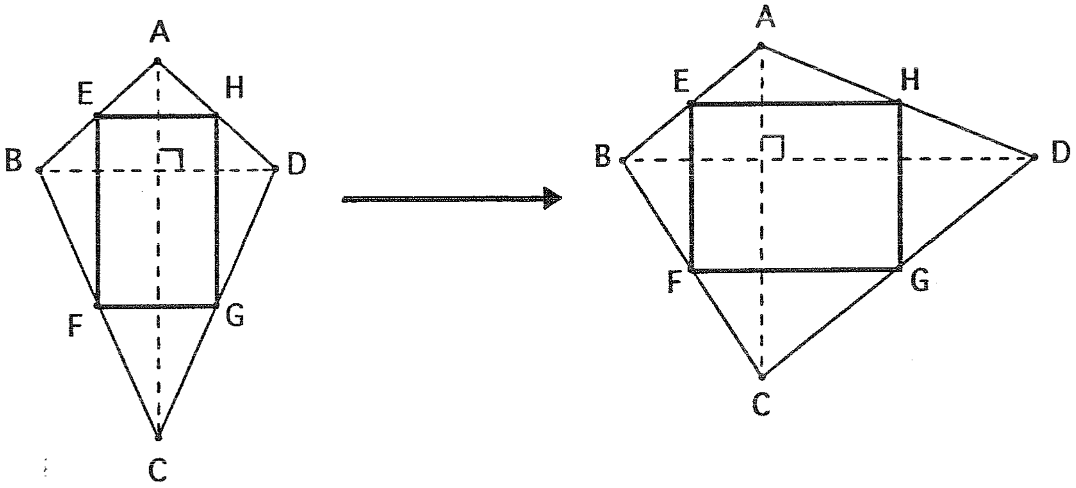


Figure 30

Proof

A deductive analysis shows that this result depends only on the *perpendicularity* of the diagonals. For example, we have EF//AC in triangle ABC and HG//AC in triangle ADC (see first figure in Figure 30). Therefore EF//HG. Similarly, EH//BD//FG and therefore EFGH is a parallelogram. Since BD ⊥ AC (property of kite) we also have for instance EF ⊥ EH, but this implies that EFGH is a rectangle (a parallelogram with a right angle is a rectangle).

Looking back

Notice that the property of equal adjacent sides (or an axis of symmetry through one pair of opposite angles) was not used at all. In other words, we can immediately **generalize** the result to a *perpendicular quad* as shown in Figure 30. (Note that it is also true for concave and crossed cases). This shows the value of understanding **why** something is true. Furthermore note that the general result was not suggested by the purely empirical verification of the original conjecture. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since most people would probably have restricted their investigation to the more familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares and rectangles. (Note that from the above proof we can also see that EFGH will always be a parallelogram in any quadrilateral).

Let's now consider the second conjecture shown in Figure 25. If we call the conjectured point of concurrency O, then it visually appears as if the six angles formed at O are each equal to 60° (see Figure 31a). By measurement and transformation on *Cabri* or *Sketchpad*, this can easily be confirmed (see Figures 31b-c). In other words, quadrilaterals ADBO, BECO and CFAO must be cyclic since the exterior angles are equal to the opposite interior angles (= 60°). We can now use this observation to produce the following proof. (Note that this illustrates another function of quasi-empirical testing and exploration, namely, assistance in the discovery/invention of a proof).

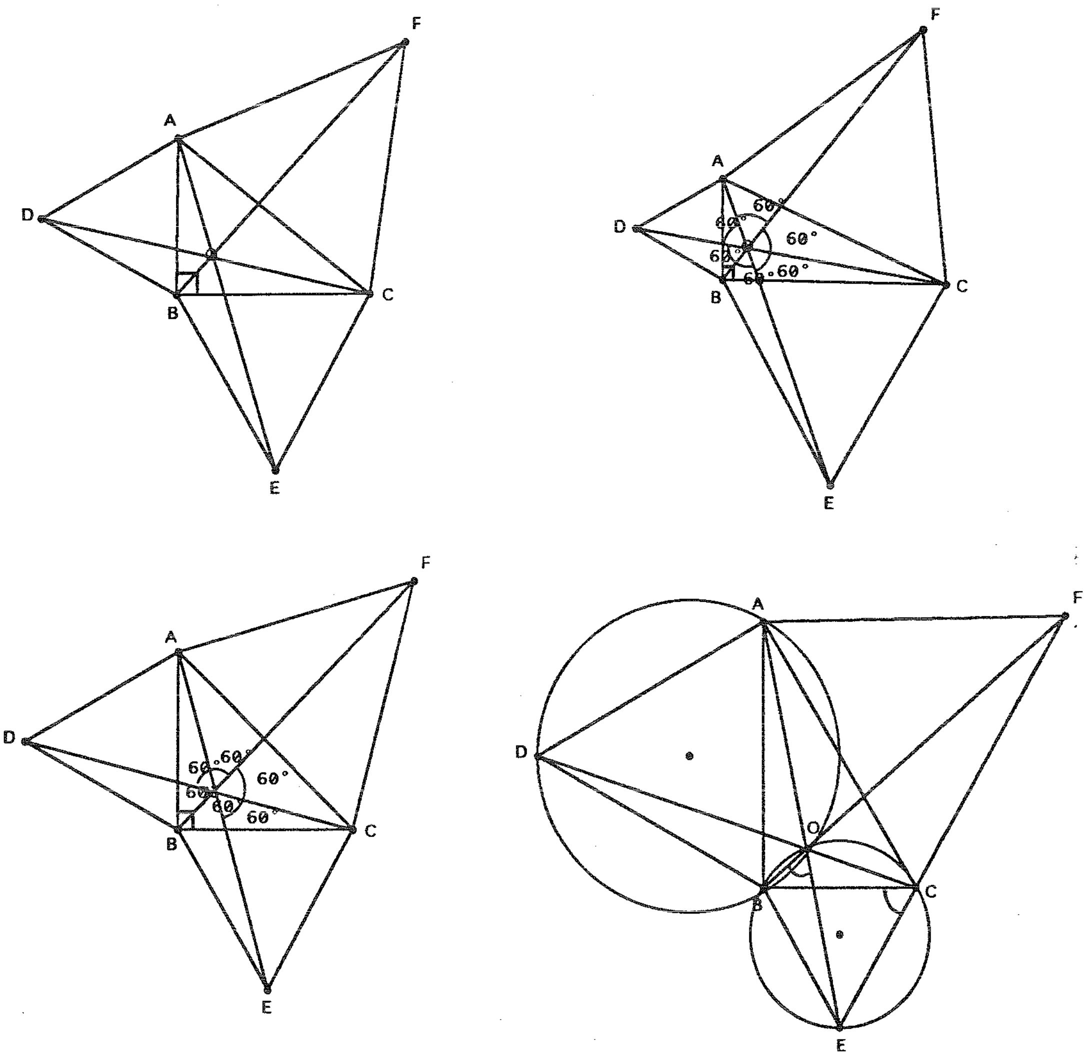


Figure 31

Proof

Construct circumcircles ADB and BEC to intersect in B and O. (See Figure 31d). Connect O with A, B, C, D, E and F. Then $\angle BOE = \angle BCE = 60°$ (inscribed angles on the same chord). But $\angle BOA = 120°$ since ADBO is cyclic. Therefore AOE is a straight line. Similarly DOC is a straight line. Also $\angle AOC = 360° - (\angle BOA + \angle BOC) = 360° - 240° = 120°$. Therefore CFAO is also cyclic and as before it follows that BOF is a straight line.

Looking back

Since we did not use the property that $\angle B = 90°$ in the above proof, it follows that this result is true for any triangle ABC. Again we see that the insight obtained from constructing a proof enables a further generalization. (It should be pointed out that the point O is normally called the Fermat point of a triangle).

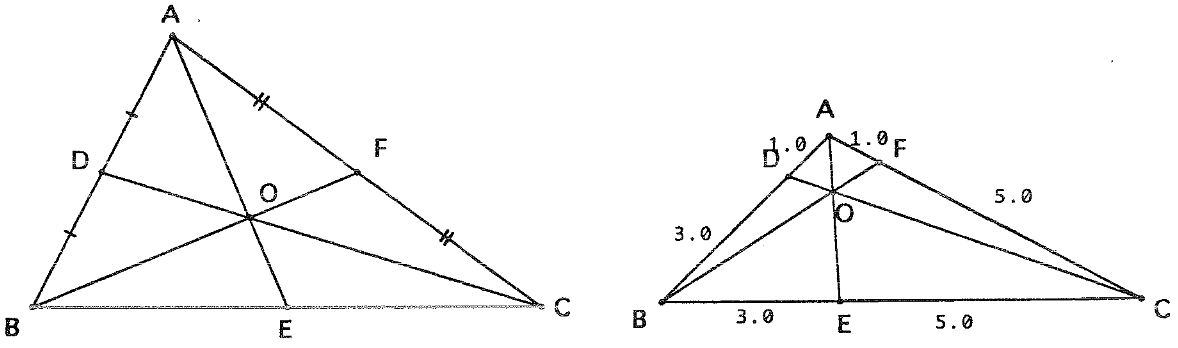


Figure 32

Another example

Suppose we want to prove the familiar result that the medians of a triangle are concurrent (see Figure 32a). Let CD and BF be medians intersecting at O. Join A with O and extend to E on BC. We now have to prove that E is the midpoint of BC. If we denote the areas of the various triangles by the following notation, area ΔABC ↔ (ABC), we have:

$$\frac{BE}{EC}=\frac{(ABE)}{(AEC)}=\frac{(OBE)}{(OCE)}=\frac{(ABE)-(OBE)}{(AEC)-(OCE)}=\frac{(ABO)}{(ACO)}$$

Similarly, we find:

$$\frac{CF}{FA}=\frac{(BCO)}{(ABO)} \text{ and } \frac{AD}{DB}=\frac{(ACO)}{(BCO)}.$$

But AD=DB and CF=FA. Therefore, (ACO) = (BCO) and (BCO) = (ABO) which implies (ACO) = (ABO). But the areas of these two triangles are proportional to BE and EC as shown by the first equation. Thus, BE/EC = 1 implies BE = EC.

Looking back

Looking back at the first part of the above proof, it is interesting to note that the **product** of the given ratios is always equal to 1, *irrespective of whether D, E and F are midpoints*, for example:

$$\frac{BE}{EC}\times\frac{CF}{FA}\times\frac{AD}{DB}=\frac{(ABO)}{(ACO)}\times\frac{(BCO)}{(ABO)}\times\frac{(ACO)}{(BCO)}=1.$$

This implies the following general result: *"If three line segments AE, BF and CD of ΔABC are concurrent, then*

$\frac{BE}{EC} \cdot \frac{CF}{FA} \cdot \frac{AD}{DB} = 1$" (see Figure 32b).

This interesting result is called Ceva's Theorem after an Italian mathematician named Giovanni Ceva who published it in 1678. In his honor the line segments AE, BF and CD joining the vertices of a triangle to any given points on the opposite sides, are called *cevians*. (Note that apart from the medians, the altitudes and angle bisectors of a triangle can be considered as cevians if extended to meet the opposite sides). Although it is not known how he discovered this result, it is likely that he discovered it in a similar fashion as outlined above, and not by using construction and measurement.

It is often popularly said by critics of the amount of deductive rigor at school level, that deduction in general (and proof in particular) is not a particularly useful heuristic device in the actual discovery/invention of new mathematical results. However as shown here, proof as a means of explanation often is a powerful heuristic for the discovery of further generalizations of results. Furthermore, there are numerous examples in the history of mathematics where new results were discovered/invented in a purely deductive manner; in fact, it is completely unlikely that some results (e.g. the non-Euclidean geometries) could ever been chanced upon merely by intuition and/or only using quasi-empirical methods. To the working mathematician proof and deductive reasoning is therefore often a means of exploration, analysis, discovery and invention (compare Schoenfeld, 1986 & De Jager, 1990).

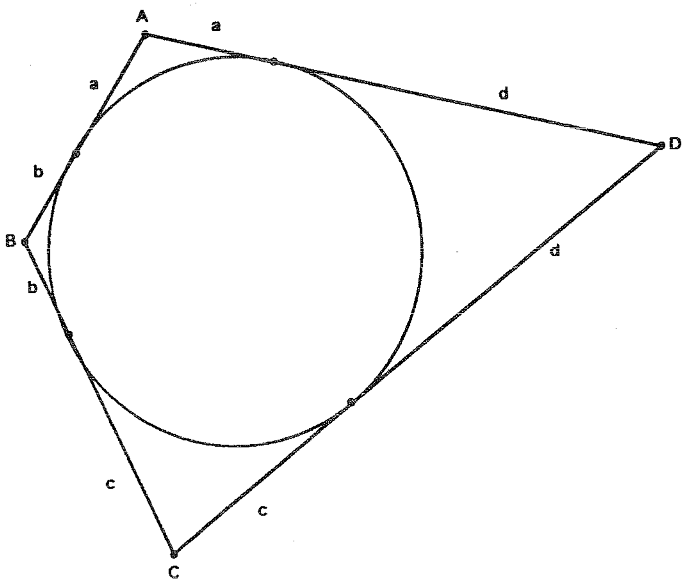


Figure 33

For example, new results can sometimes be discovered by simply deductively analysing the properties of given objects. Let's suppose for instance that we want to investigate the properties of a *circum quad* (a quadrilateral circumscribed around a circle) as shown in Figure 33. Assuming one already knows that the tangents to a circle from a point outside the circle are equal, one might proceed by indicating their equality as shown. If one now looks carefully at this diagram, the observant reader might immediately notice without *actually measuring* that AB + CD = $a + b + c + d$ = BC + AD. In other words, that the sums of the opposite sides of a circum quad are equal.

Apart from the traditionally stressed function of proof as a means of verification, as well as of explanation and discovery as briefly outlined here, proof also fulfills other functions such as *systematization* and *communication.* (For more details, consult De Villiers, 1990). Lastly, it is important to point out that proving something is an intellectual challenge which some people find as appealing as other people may find puzzles or other creative hobbies. It could for instance be compared to the physical challenge of completing an arduous marathon or triathlon. In this sense, proof serves the function of *self-realization* and *fulfillment.* Proof is therefore a testing ground for the stamina and ingenuity of the mathematician. To paraphrase Sir Edmund Hilary's famous comment on his reason for climbing Mount Everest: *"we prove our results because they're there."*

Making conjectures

At the heart of making conjectures lies the ability to look and ask *questions* from different perspectives. For example, it is a good habit to acquire to ask oneself the following questions when one comes across any mathematical problem, result or situation since it may lead to the formulation of new conjectures:

- What if ... is changed?
- What happens if ...?
- What if ... not?
- What could be a generalization?
- What could be an analogous variation?

Although the investigation of such questions do not necessarily lead to entirely new and exciting results, one will occasionally be successful if one continually nurtures a questioning habit. As an example let's consider the well-known Pythagorean theorem. Basically this theorem can be interpreted in at least three ways as follows:

- *algebraically* : If c is a hypotenuse of a right triangle, and a and b are the other two sides, then $a^2 + b^2 = c^2$.
- *geometrically* : The area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the other sides (see Figure 34).
- *analytically* : The distance d between two points with coordinates (x_1, y_1) and (x_2, y_2) is determined by $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

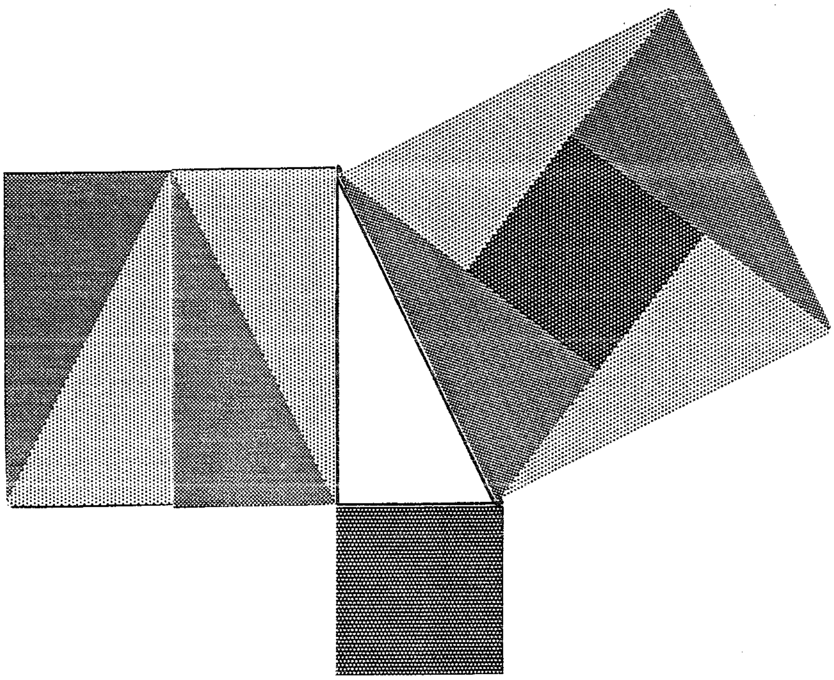


Figure 34

Examples of some questions that could lead to rich mathematical investigations of the Pythagorean theorem are:

(a) What would be the relationship between the sides if we did not have a right triangle, but *any triangle*?

(b) What if we did not have squares but *other regular polygons* on the sides of a right triangle?

(c) Could we extend the result into *three dimensions* by relating the *volumes* of certain solids to each other?

(d) What if we did not have a right triangle, but a *rectangle*?

(e) Is it possible in *any triangle* to find a number p such that $c^p = a^p + b^p$?

(f) Another interpretation of the Pythagorean theorem is that it determines the length of the diagonal of a rectangle. Can we extend this interpretation into three dimensions by relating the *diagonal of a solid* to its sides?

(g) What would be the distance d between two points in *n-dimensional space* with coordinates $(p_1, p_2, \ldots p_n)$ and $(q_1, q_2, \ldots q_n)$?

(h) Are there analogous versions of the Pythagorean theorem in the *non-Euclidean geometries*?

The false impression is unfortunately sometimes created that mathematicians are only problem solvers who try to solve *already given* problems. However, mathematicians continually create their own new problems by asking questions, making hypotheses and testing them. Mathematicians are therefore not mere problem-solvers, but also creative problem posers. As Hilton (1989:15) put it:

"Our mathematical knowledge ... does more than enable us to solve already formulated problems; it tells us ***how*** *to formulate questions,* ***what*** *questions to ask, and* ***when*** *to ask them."*

Polya (1973:205-206) similarly stresses that *"the future mathematician should be a clever problem-solver; but to be a clever problem-solver is not enough ... the most important part of the work is to look back at the completed solution. Surveying the course of his work and the final shape of the solution, he may find an* ***unending variety of things to observe*** *... He should solve problems, ... meditate upon their solution, and* ***invent new problems****"*. (bold added).

Finally, it should be remembered that mathematics does not start with axioms, definitions and theorems, but starts in *questions*, develops in a *quasi-empirical* fashion and only ends in axioms, definitions and theorems.

Questions and Problems 3

(Suggestion: The reader is strongly advised to carry out the following investigations with dynamic software like *Cabri Géomètre* or *Geometer's Sketchpad*).

1. How would you try and generalize the result shown in Figure 31 regarding the Fermat point of a triangle? Investigate.
2. Investigate the questions in the previous section regarding the Pythagorean theorem.
3. Is the converse of the theorem of Ceva (Figure 32) true or not? If so, provide a proof; if not, a counter-example.
4. (a) Can a circum quad be concave or crossed? Investigate.
   (b) Formulate the converse of the result for circum quads shown in Figure 33. Is it true or not? If so, provide a proof; if not, a counter-example.
5. Consider the earlier mentioned result: If the midpoints E, F, G and H of the sides of any quadrilateral ABCD are consecutively connected, then EFGH is a parallelogram (see Figure 35). How would you try to generalize this result? Investigate.

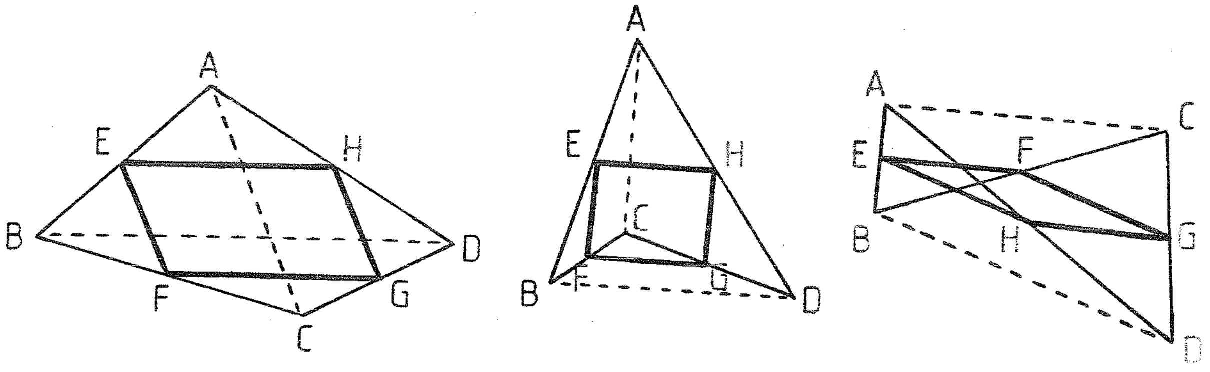


Figure 35

6. Noble (1990:168-170) describes the useful apparatus shown in Figure 36 with which

to dynamically illustrate the previous theorem. Keeping AB fixed and moving for instance D one can see that EFGH remains a parallelogram no matter how the quadrilateral changes shape. He also makes the following interesting observation and conjecture:

"... it can be observed that G appears to move in a circular arc with E as its center and a diagonal of the parallelogram, EG, as its radius. If this observation is correct then another interesting property emerges, namely: For any quadrilateral of given sides, the diagonals of the parallelogram formed by joining the mid-points of the sides of the quadrilateral, are of constant lengths."

Is this really true? If so, can you explain it? If not, can you provide a counter-example?

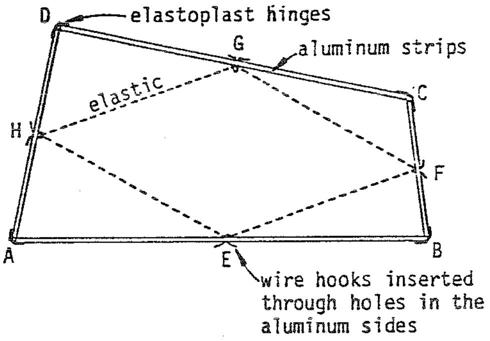


Figure 36

7. Connect the circumcentres of the equilateral triangles constructed on the sides of the triangle in Figure 37 and draw the circumcircles. What do you notice? Is it always true? If so, can you explain your observation? If not, can you provide a counter-example?

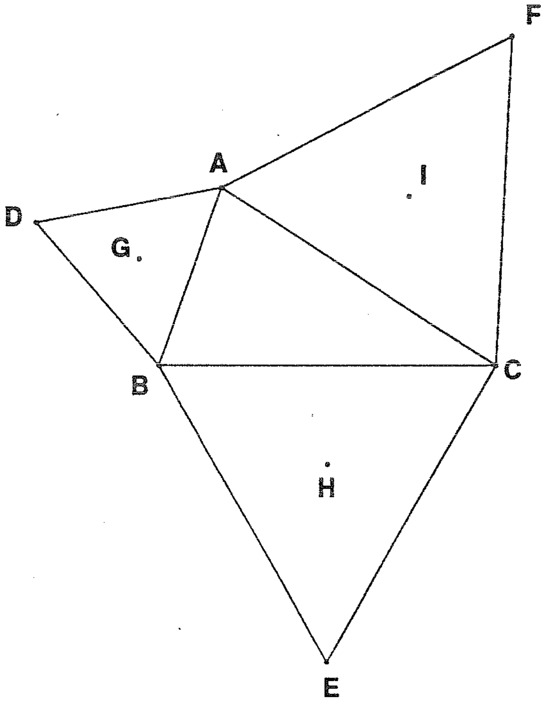


Figure 37

8. Connect the three points of intersection X, Y and Z of the adjacent trisectors of the

angles of the triangle shown in Figure 38. What do you notice? Do you think it is always true? Investigate.

9. Connect the centres of the squares constructed on the sides of the parallelogram as shown in Figure 39. What do you notice? Is it always true? Investigate.

10. Figure 40 shows a convex quadrilateral ABCD with equilateral triangles ABM_1, BCM_2, CDM_3 and DAM_4 constructed on the sides so that the first and third are exterior to the quadrilateral, while the second and the fourth are on the same side of sides BC and DA as is the quadrilateral itself. Connect M_1, M_2, M_3 and M_4. What do you notice? Is it always true? Investigate.

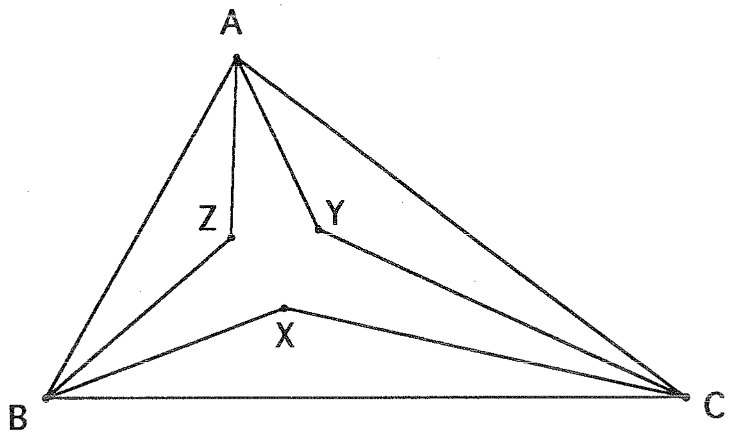


Figure 38

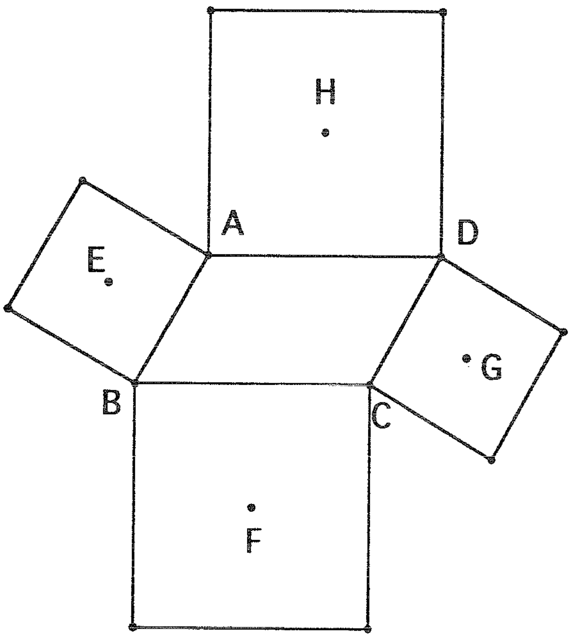


Figure 39

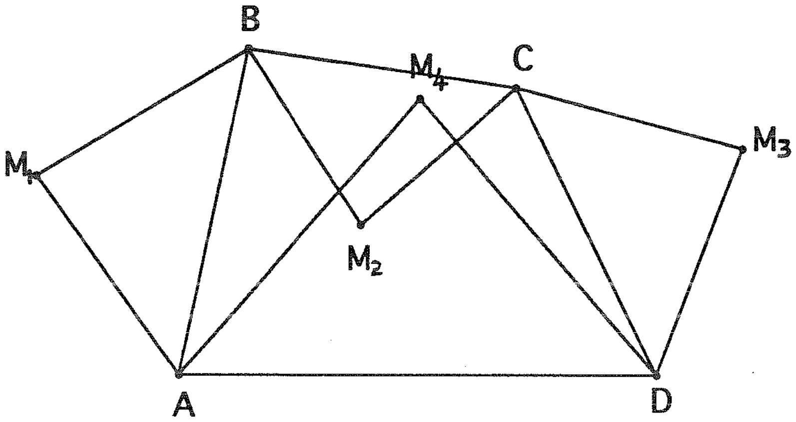


Figure 40

Figures 41 - 42 show convex quadrilaterals ABCD with AD = BC and AD inclined towards BC at 60° to each other. (The latter condition might also be stated in the form $\angle A + \angle B = 120°$).

11. In Figure 41 connect the indicated midpoints P, Q and R of the diagonals and the side CD. What do you notice? Is it always true? Investigate.

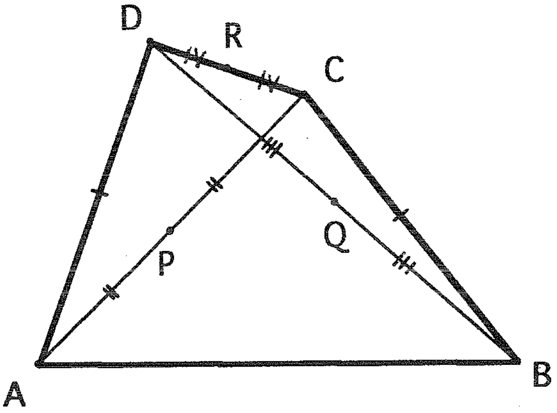


Figure 41

12. In Figure 42 an equilateral triangle PCD has been drawn outwardly on CD. What can you say about triangle PAB? Is it always true? Investigate.
13. In Figure 43 equilateral triangles have been drawn on AC, DC and DB. What do you notice about points P, Q and R? Is it always true? Investigate.
14. If not already done so, try asking *"What if?"* questions for all the preceding questions, and investigate further.

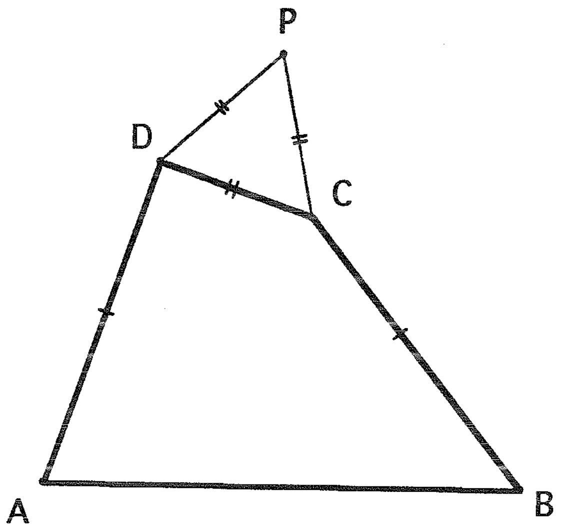


Figure 42

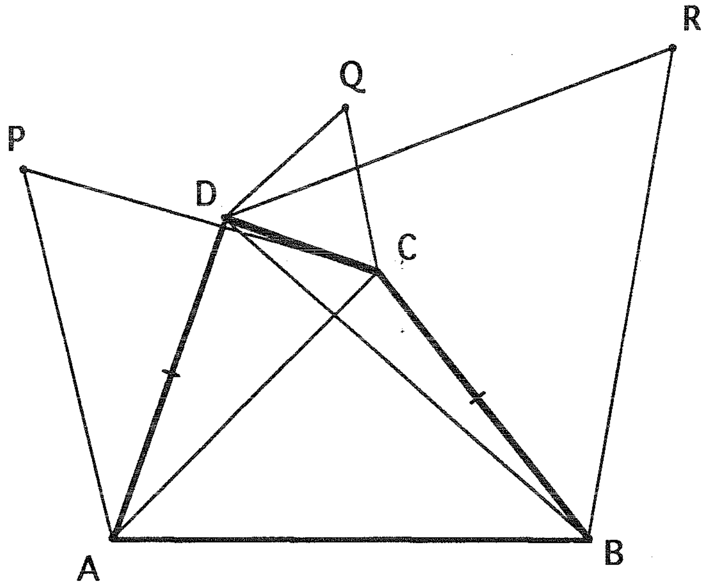
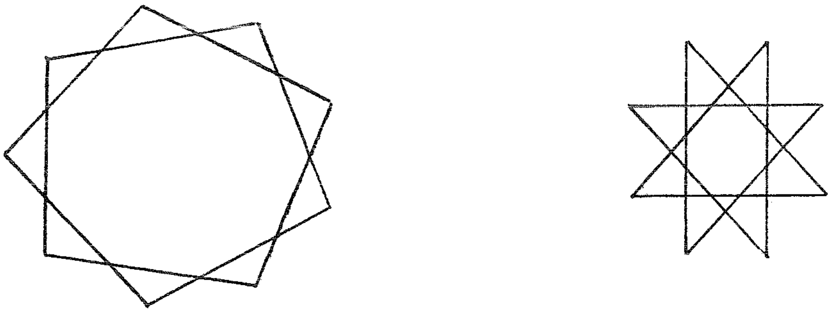


Figure 43

15. The figures in Figure 44 were created with the recursive LOGO procedure: TO POLY. It has two variables as input, the first one being the *length* and the second the *turning angle*.
(a) What can you say about the sum of the internal angles of each of the regular figures in Figure 44?
(b) What happens to the sum of the internal angles if these figures are not regular as shown in Figure 45?

POLY 80 144

POLY 80 160



POLY 80 80

POLY 80 135

Figure 44

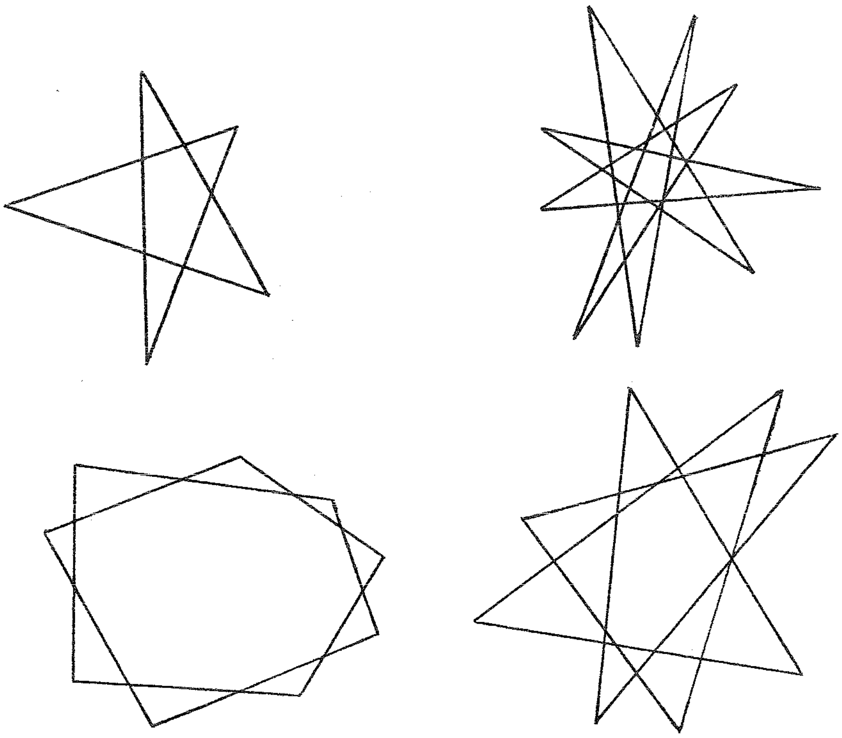


Figure 45

16. Critically assess, and reformulate if necessary, the following well-known theorems from school:

(a) All cyclic quadrilaterals have opposite angles supplementary.

(b) A quadrilateral with opposite angles supplementary is cyclic.

(c) The sum of the internal angles of any poygon ($n > 2$) is $(n-2)180°$.

17. What properties of a parallelogram (besides opposite sides parallel and equal) does a parallelo-$2n$-gon in general have? (See *Questions and problems 2, No.17*, Figure 2.34). Investigate.

Chapter 4

An interesting duality

"Symmetry as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend, and create order, beauty and perfection." - Hermann Weyl

Duality is a special kind of symmetry. In everyday language, a common duality exists between antonyms such as hot and cold, tall and short, love and hatred, male and female, etc. Basically, the one concept is defined by and understood in terms of the other, and together they form a whole which complement and enrich each other.

In mathematics there are often similar dualities between certain concepts and operators. For example, in projective geometry we find an interesting duality between the following concepts:

vertices (points) - sides (lines)
inscribed in a circle - circumscribed around a circle
collinear - concurrent

Two theorems or configurations are called *dual* if the one may be obtained from the other by replacing each concept and operator by its dual concept or operator. This duality is strikingly reflected by the projective theorems of Pascal (1623 - 1662) and Brianchon (1785 - 1864) as follows (see Coxeter & Greitzer, 1967:74-78 or Yaglom, 1973:181-183 for proofs):

Pascal's theorem

If a hexagon is *inscribed* in a circle, then the three *points of intersection of the opposite sides* are *collinear* (lie in a straight line) (Figure 46a).

Brianchon's theorem

If a hexagon is *circumscribed* around a circle, then the three *lines* (the diagonals) *connecting opposite vertices* are *concurrent* (meet in the same point) (Figure 46b).

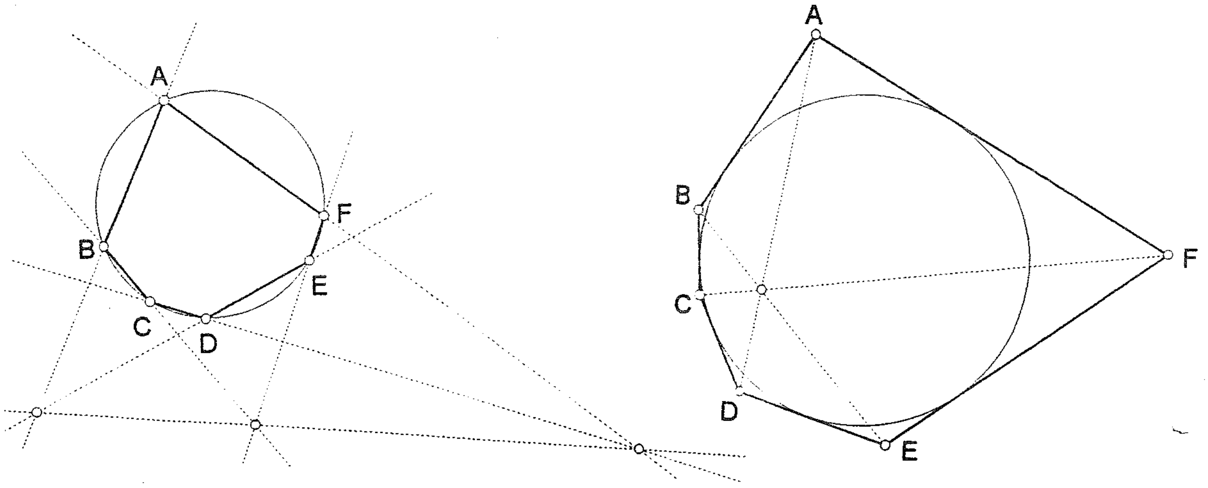


Figure 46

In fact, not only the theorems of Pascal and Brianchon, but *all* the theorems of projective geometry occur in such pairs, each similar to the other and identical in structure, except for the *interchange* of dual concepts. In projective geometry therefore the *dual* of any true theorem, is another true theorem. In fact, it is unnecessary to prove the dual results since their proofs can be obtained by simply writing down the proofs of the original results word by word, replacing only relevant concepts with their corresponding duals. This is of course only true if it has already been established that such a duality exists in general (as has been done in projective geometry).

Duality in triangles

Interestingly, there exists a similar, although limited duality between the concepts *angle* (vertex or point) and *side* (linesegment) within Euclidean plane geometry which we will explore in this chapter. For example, consider the following well-known theorems:

1. If a triangle has two sides equal, then it has two equal angles.
2. If a triangle has two angles equal, then it has two equal sides.
3. If a triangle has all its sides equal, then all its angles are equal.
4. If a triangle has all its angles equal, then all its sides are equal.

Here we clearly have (1) and (2), and (3) and (4), as each other's duals, since the replacement of the concept *"sides"* by its dual *"angles"*, and vice versa, transforms the one into the other. However, this duality is not true in general. For example, the sum of the angles of a triangle is constant, but the sum of the sides of a triangle is clearly variable, depending for instance on the size of the triangle.

There is a similar duality between the concepts *"angle bisector"* and *"perpendicular bisector"* which can be formulated as follows:

An *angle bisector* is the locus of all the points equidistant from the two *sides* of an *angle* (see Figure 47a).

A *perpendicular bisector* is the locus of all the points equidistant from the two *endpoints* of a *line segment* (side) (see Figure 47b).

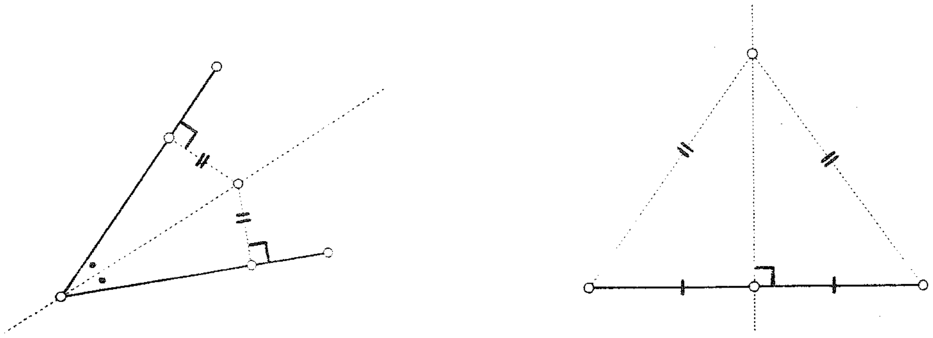


Figure 47

The following two theorems involving these concepts are therefore also dual:

The *angle bisectors* of the *angles* of any triangle are concurrent at its *incentre* (the centre of the inscribed circle).

The *perpendicular bisectors* of the *sides* of any triangle are concurrent at its *circumcentre* (the centre of the circumscribed circle).

These two theorems can furthermore be generalised to any polygon as follows:

The *angle bisectors* of any *circum polygon* (a polygon circumscribed around a circle) are concurrent at the *incentre* of the polygon (e.g. see Figure 48a which shows a circum quad).

The *perpendicular bisectors* of any *cyclic polygon* are concurrent at the *circumcentre* of the polygon (e.g. see Figure 48b which shows a cyclic quad).

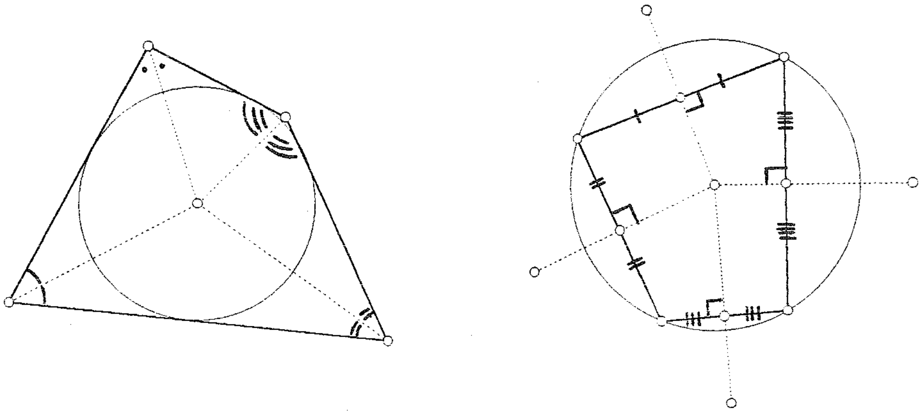


Figure 48

The above two generalizations of course follow automatically from the requirement that the incentre or circumcentre of a polygon (if they exist) must respectively be equidistant from the sides or the vertices. The following two examples also reflect the duality between incentres and circumcentres:

Any triangle that has two equal *angle bisectors* (each measured from a vertex to the *incentre*) is isosceles (Figure 49a).

Any triangle that has two equal *perpendicular bisectors* (each measured from its corresponding midpoint to the *circumcentre*) is isosceles (Figure 49b).

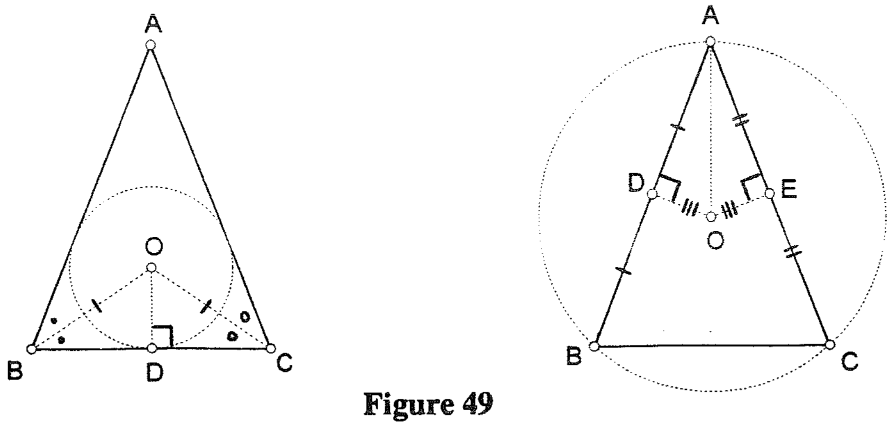


Figure 49

Duality in quadrilaterals

The same duality between angle and side is also reflected in different types of quadrilaterals. For example consider the following:

Square

All *angles* equal	All *sides* equal
Circumscribed circle (*cyclic*)	Inscribed circle (*circum* quad)
An axis of symmetry through each pair of opposite *sides*	An axis of symmetry through each pair of opposite *angles*

Rectangle	**Rhombus**
All *angles* equal	All *sides* equal
Circumscribed circle (*cyclic*)	Inscribed circle (*circum* quad)
An axis of symmetry through each pair of opposite *sides*	An axis of symmetry through each pair of opposite *angles*

Isosceles trapezium	**Kite**
Two pairs of equal adjacent *angles*	Two pairs of equal adjacent *sides*
One pair of equal opposite *sides*	One pair of equal opposite *angles*
Circumscribed circle (*cyclic*)	Inscribed circle (*circum* quad)
An axis of symmetry through one pair of opposite *sides*	An axis of symmetry through one pair of opposite *angles*

Cyclic quad	**Circum quad**
Circumscribed circle (*cyclic*)	Inscribed circle (*circum*)
Perpendicular bisectors of the *sides* are concurrent at the *circumcentre*	*Angle* bisectors of the *angles* are concurrent at the *incentre*
The sums of the two pairs of opposite *angles* are equal (e.g. $\angle A+\angle C=\angle B+\angle D=180°$ for convex; $=360°$ for crossed)	The sums of the two pairs of opposite *sides* are equal (e.g. AB + CD = BC + AD) (see Figure 33)

Parallelogram

Equal opposite *angles*	Equal opposite *sides*
The two distances from the point of symmetry to any pair of opposite *angles* are equal (see Figure 50a).	The two distances from the point of symmetry to any pair of opposite sides are *equal* (see Figure 50b).

(Also note that a general parallelogram is neither cyclic nor a circum quad).

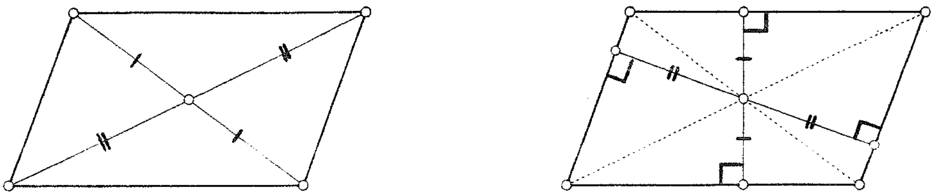


Figure 50

From the above,we can clearly see that rectangles and rhombi, isosceles trapezia and kites, and cyclic and circum quads are each other's duals. On the other hand, the squares and parallelograms are their own duals; in other words, *self-dual*.

Utilizing duality in the discovery of new results

In 1840, the following elementary looking result was mentioned in a letter from Lehmus to Sturm, with a request for a pure geometric proof rather than the one Lehmus had constructed himself:
"Any triangle that has two equal angle bisectors (each measured from a vertex to the opposite side) is isosceles."

Sturm mentioned it to a number of mathematicians, and one of the first to prove it geometrically was the great Swiss geometer Jacob Steiner, and it consequently became known as the Steiner-Lehmus theorem. Papers on it appeared in various journals in 1842, 1844, 1848, almost every year from 1854 until 1864, and with a good deal of regularity during the next hundred years or so. What makes this result interesting is that it looks like a routine exercise from high school, but is not quite that simple.

Here is a proof which was published in the *Mathematical Gazette* in 1982, but was later discovered to be identical to a proof published in 1880 (from Webb, 1987: 41).

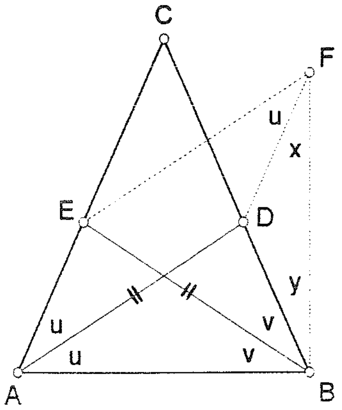


Figure 51

Consider Figure 51 with AD and BE the equally given bisectors. Construct the parallelogram AEFD and join FB. In the triangle BEF, BE(= AD) = EF and so $\angle EFB = u + x = v + y = \angle FBE \ldots (1)$.

Now if u and v are different, we may assume that one is greater. Suppose that $u > v$. Then BD > EA, and hence BD > FD. Therefore $x > y$ and adding it to the previous inequality involving u and v produces: $u + x > v + y$ ... (2).

Statements (1) and (2) are clearly contradictory, and we are forced to the conclusion that u cannot be greater than v. A similar argument shows that v cannot be greater than u, and we have to conclude that $u = v$, and that the triangle is isosceles.

An interesting aspect of the Steiner-Lehmus theorem is that everyone of the hundreds of proofs so far devised relies on the principle of contradiction. In 1882, the famous mathematician J J Sylvester argued that a direct proof is simply not possible, but he did not give a satisfactory justification for his statement.

Using the afore-mentioned duality between angle and perpendicular bisectors we can conjecture the following dual to the Steiner-Lehmus theorem:

> "Any triangle that has two equal *perpendicular* bisectors (each measured from a midpoint to the opposite side) is isosceles."

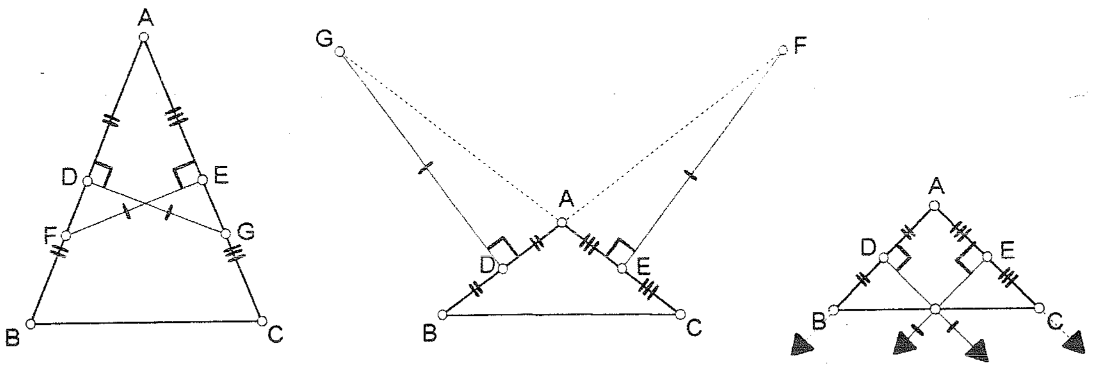


Figure 52

An investigation shows that it is indeed true. Consider Figure 52. The proof in the first two cases is quite simple since triangles AEF and ADG are congruent ($\angle, \angle, s$), and therefore AB = AC. The proof of the third case can be conceptualized in terms of projective geometry where parallel lines intersect at infinity, at the so-called *vanishing point.* Since the perpendiculars through D and E are respectively parallel to sides AC and AB we can conceptualize that they meet in the respective vanishing points G' and F'. However, then as before, we have *"triangles"* AEF' and ADG' congruent ($\angle, \angle, s$). (Is the preceding proof still valid if G and F respectively fall on AC and AB extended?).

What about a duality between the *diagonal* properties of the respective quadrilaterals? If we compare the diagonal properties of rectangles and isosceles trapezia respectively with those of rhombi and kites, it appears that **equal diagonals** and **perpendicular diagonals** are each others duals. If we now consider the result shown in Figure 24 and its generalization in Figure 30, we can immediately conjecture that connecting the midpoints of the adjacent sides of an isosceles trapezium and a *diagonal* quad (a quadrilateral with equal diagonals) would produce the dual of a rectangle, namely a rhombus. This duality between a diagonal and perpendicular quad is illustrated in Figure 53 (and is left to the reader to explain).

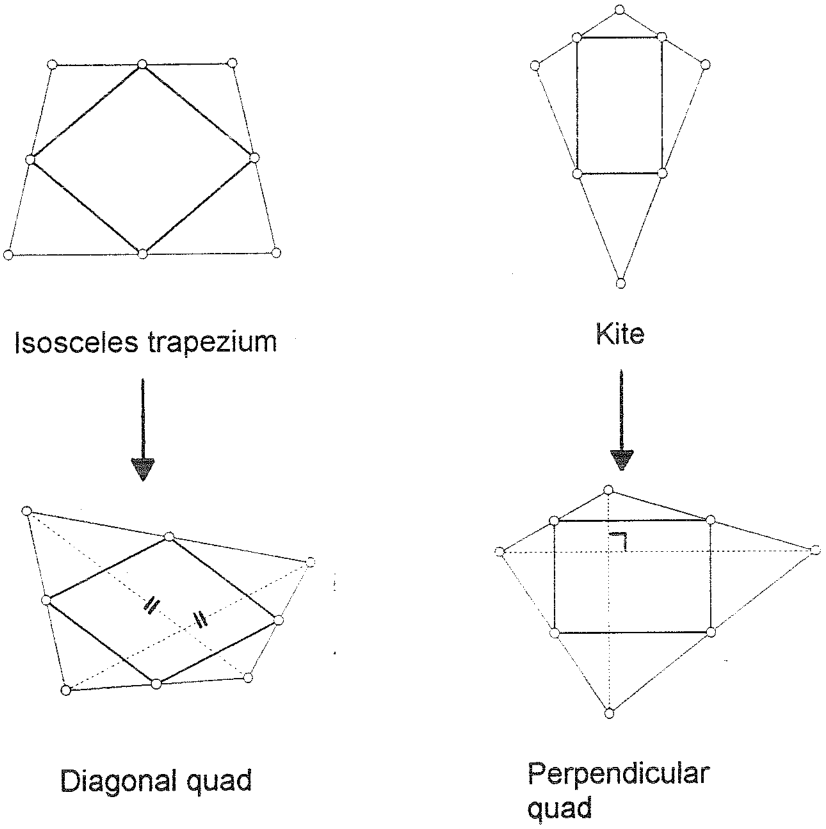


Figure 53

Let's consider one more example. Suppose we have a convex circum quad with side lengths of a, b, c and d as shown in Figure 54a. Select *any* point P on AB. Take Q on BC so that BQ = PB, R on CD so that CR = QC and S on AD so that DS = RD. Then we have the surprising result that AS = AP and PQRS is a cyclic quadrilateral. (Compare Van Duyn, 1987). (Note that this result is a generalization of the result that the four points where the incircle touches the sides of a circum quad are concyclic).

Proof

Let PA = x, then PB = $a - x$ = BQ, QC = $b - a + x$ = CR and RD = $c - b + a - x = (a + c) - b - x = (b + d) - b - x = d - x$ = SD ... ($a + c = b + d$ for a circum quad). Then AS = $d - (d - x) = x$.

From the above we now have triangles APS, BPQ, CQD and DRS isosceles so that $\angle APS = \angle ASP = p, \angle BPQ = \angle BQP = q, \angle CQR = \angle CRQ = r$ and $\angle DRS = \angle DSR = s$.

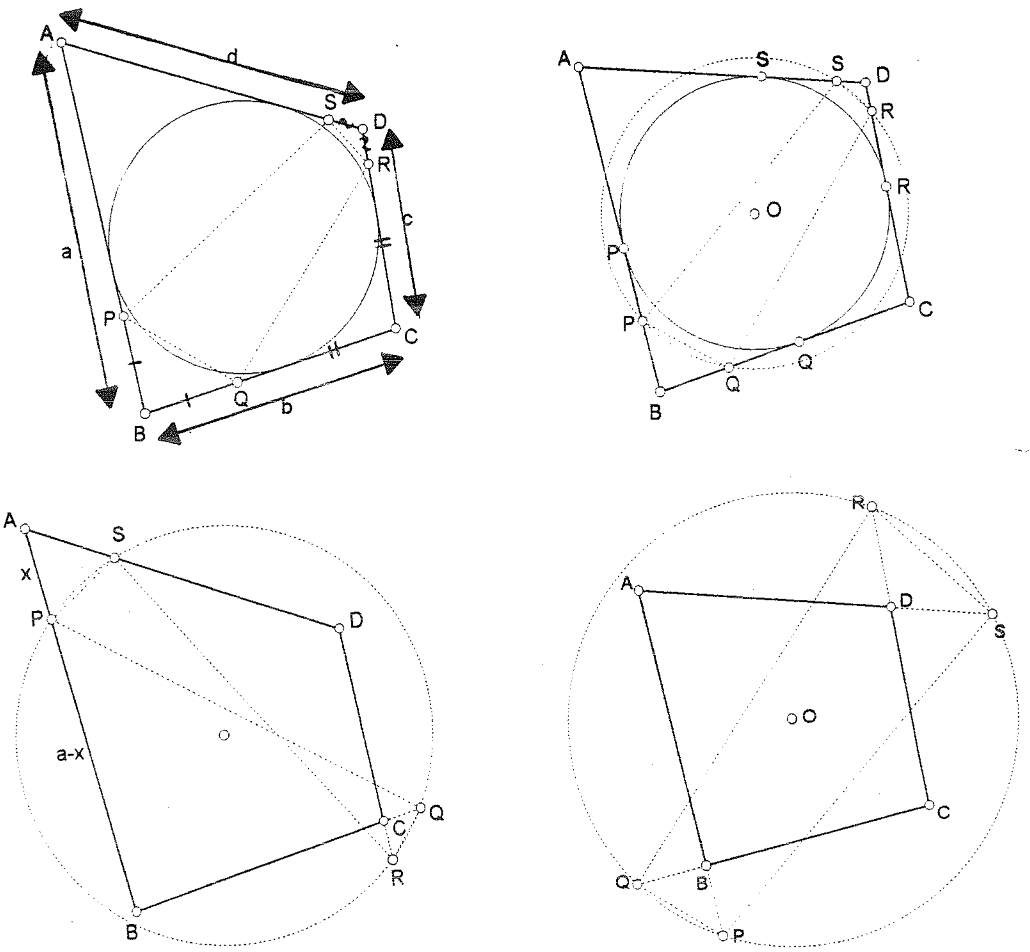


Figure 54

Therefore $\angle A = 180° - 2p, \angle B = 180° - 2q$, et cetera, and $\angle A + \angle B + \angle C + \angle D = 720° - 2(p+q+r+s)$. But $\angle A + \angle B + \angle C + \angle D = 360°$ and therefore $p+q+r+s = 180°$. But $\angle SPQ = 180° - (p+q)$ and $\angle QRS = 180° - (r+s)$. Therefore $\angle SPQ + \angle QRS = 360° - (p+q+r+s) = 180°$ from which it follows that PQRS is cyclic.

But what happens if say Q falls on the extension of BC (e.g. when PB > BC)? Is the result still true?

Further investigation/looking back

1. Since all isosceles triangles APS for different choices of P are obviously similar, it follows that all their sides PS must have the same perpendicular bisector (axis of symmetry). Since this is also true for the other isosceles triangles, we have that the circumcentre of the cyclic quad PQRS is fixed; or in other words, that all the circles PQRS are *concentric* with the incircle of ABCD (see Figure 54b).

2. It is interesting to note that the points P, Q, R and S need not necessarily fall on the sides AB, BC, CD and AD. If for example Q falls on the extension of BC so that QC = *a* - *x* - *b*, we need only construct CR on the extension of DC as shown in Figure 54c to

maintain the result. In fact, as shown in Figure 54d none of the points need fall on the sides.

Again using the aforementioned dualities we can now conjecture the following dual to the previous result:

"Construct any *angle divider* AQ of $\angle A$ of a convex *cyclic quad* ABCD. Now construct the angle divider BS of $\angle B$ so that $\angle PBA = \angle PAB$, the angle divider CR of $\angle C$ so that $\angle SCB = \angle SBC$ and the angle divider DQ of $\angle D$ so that $\angle RDC = \angle RCD$ (see Figure 55). Then $\angle QDA = \angle QAD$ and PQRS is a *circum quad.*"

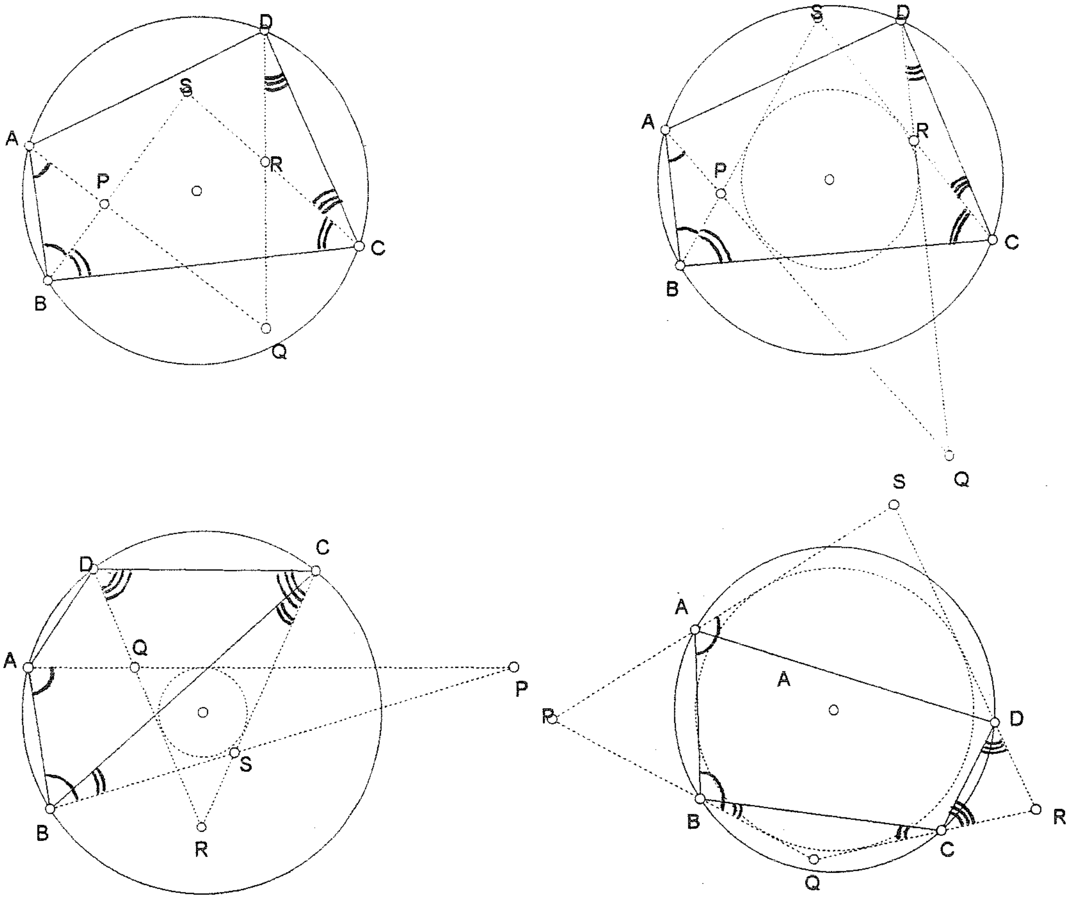


Figure 55

Proof

Let $\angle PAB = x$, then $\angle PBA = x$, $\angle SBC = \angle B - x = \angle SCB$, $\angle RCD = \angle C + x - \angle B = \angle RDC$ and $\angle QDA = \angle D - \angle C - x + \angle B = (\angle D - \angle C + \angle B) - x = \angle A - x = \angle QAD$.

From the equality of the angles, we have QA = QD, PA = PB, SB = SC and RC = RD. Therefore PS + QR = (SB - PB) + (QD - RD) = SB + QD - BP - RD and PQ + RS = (QA - PA) + (SC - RC) = SC + QA - PA - RC = SB + QD - BP - RD. Therefore PQRS is a circum quad (the sums of the opposite sides are equal - see *Questions & Problems* 2, no.4).

As with the previous case, will the result still be true if the angle dividers fall outside the angles? Are all the circles PQRS also concentric with the circumcircle?

Further investigation/looking back

1. Since all isosceles triangles PAB for different choices of AQ obviously have the same axis of symmetry through P, it follows that the angle bisector of P is fixed. Since this is also true for the other isosceles triangles, we have that the incentre of the circum quad is fixed; or in other words, that all the circles PQRS are concentric with the circumcircle of ABCD (see Figures 55b - d).

2. It is interesting to note that the angle dividers AQ, BS, CR and DQ need not necessarily be "*inside* " the angles A, B, C and D. If for example BSP falls outside angle B so that $\angle SBC = x - \angle B$, we need only construct CR as shown in Figure 55c to maintain the result. In fact, as shown in Figure 55d none of the angle dividers need fall inside the angles.

As shown above, we can therefore conveniently use this (limited) duality in plane geometry to conjecture new dual theorems for existing ones. However, unlike a true duality (like that in projective geometry or Boolean algebra) it is necessary to carefully check one's conjecture(s) by construction and measurement, as well as proof.

Questions and Problems 4

(Suggestion: the reader is strongly advised to carry out the following investigations with dynamic software like *Cabri Géomètre* or *Geometer's Sketchpad*).

1. What would be the duals of the following concepts? (See Chapter 1). Is there a duality between their properties?
   (a) right kite
   (b) trilateral trapezium
   (c) skew kite
   (d) skew cyclic quad

2. Consider Question 17 from *Questions and Problems 2* (see Figure 2.30). Which of the quadrilateral generalizations of a parallelogram are dual? Is there a duality between their properties?

3. Consider different quadrilateral generalizations of an isosceles trapezium. Which of these are dual to the generalizations of a kite shown in Figure 10 (Chapter 2)? Is there a duality between their properties?

4. In *Questions and Problems 2*, no.14, it was shown that a cyclic quad with equal diagonals is always an isosceles trapezium. Formulate a dual and investigate whether it is true or not.

5. Draw a classification scheme involving as many of the preceding quadrilaterals as you can to illustrate the duality between them. (Also include concave and crossed quadrilaterals).

6. (a) Prove that if the diagonals of a circum quad intersect at its incentre, then it is a rhombus.
(b) Formulate a dual of this result and investigate whether it is true or not.

7. (a) Figure 56 shows O the incentre of triangle ABC and the circumcentre P of triangle BOC. What do you notice about the points A, O and P? Is it always true? If so, can you explain why it is true? If not, can you provide a counter-example?
(b) Formulate a dual for your observation above and investigate whether it is true or not.

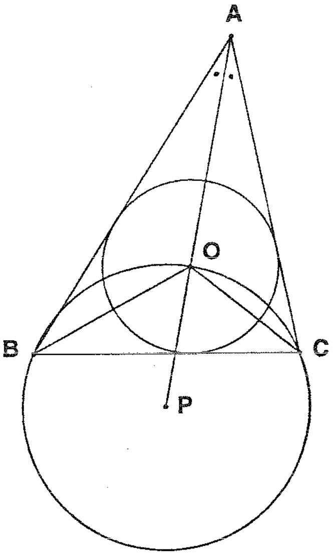


Figure 56

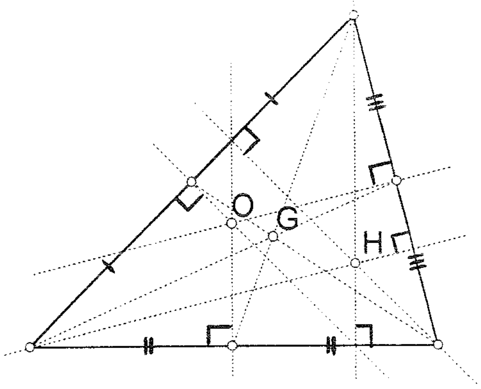


Figure 57

8. (a) Figure 57 shows the orthocentre H, centroid G and circumcentre O of a triangle. What do you notice about these three points? Is it always true? Investigate.
(b) Formulate a dual for your observation in (a) and investigate whether it is true or not.

9. (a) Prove for any (simple closed) skew kite ABCD with AB = AD, that if P and S are the midpoints of sides AD and DC, and Q and R are the respective midpoints of the diagonals AC and BD, then PQRS is a diagonal quad (see Figure 58).
(b) Formulate a dual and investigate whether it is true or not.

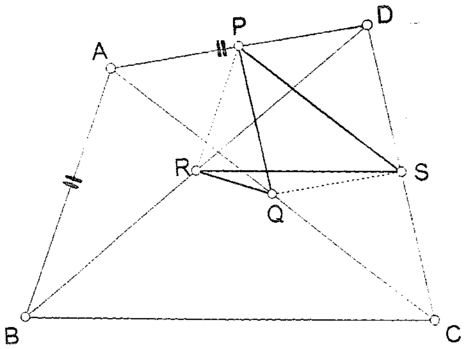


Figure 58

10. (a) Prove the following two results:

    (i) If similar isosceles triangles DBA, ECB and FAC are erected on the sides of any ΔABC, then the line segments (GC, HA and IB) connecting their circumcentres with the opposite vertices of the base triangle are concurrent (see Figure 59).

    (ii) If similar triangles ABD, EBC and AFC are erected on the sides on any ΔABC, then the line segments connecting the incentres with the opposite vertices of the base triangle are concurrent (see Figure 60).

    (b) Formulate duals for the above results and investigate whether they are true or not.

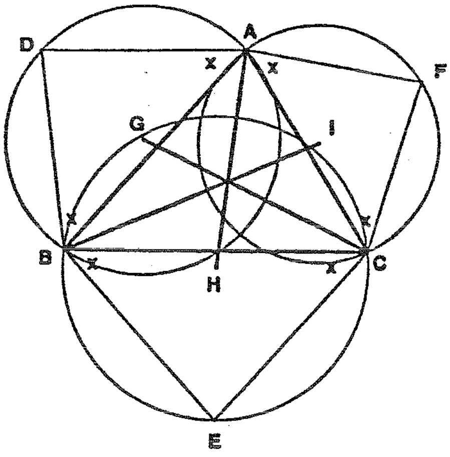


Figure 59

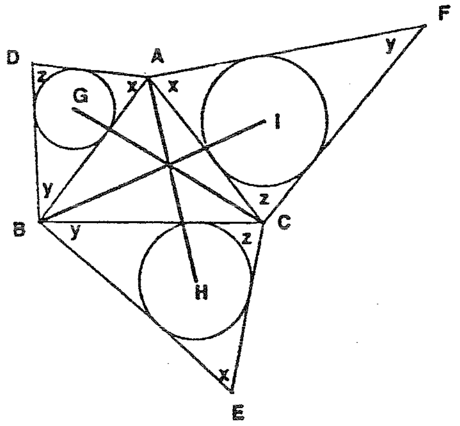


Figure 60

11. (a) Consider Figure 61. Draw the diagonals of the circum quad ABCD and those of its associated cyclic quad PQRS. What do you notice? Investigate whether it is always true.

    (b) Formulate a dual and investigate whether it is true or not.

12. Formulate a dual to Morley's theorem (see *Questions and Problems 3*, no.8) and investigate whether it is true or not.

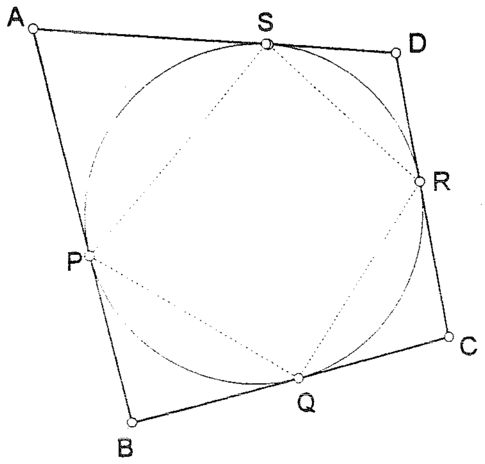


Figure 61

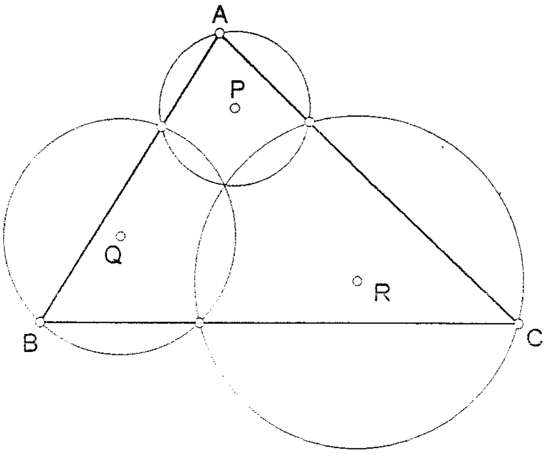


Figure 62

13. (a) Consider Figure 62 which shows a triangle with circles drawn through each vertex and the indicated points on adjacent sides. What do you notice? Investigate further.

(b) Connect the circumcentres of the three circles. What do you notice about this triangle? Investigate further.

(c) Formulate a dual for your observation in (b) and investigate whether it is true or not.

14. Formulate duals for Napoleon's theorem and its generalizations (see *Questions and Problems 3*, no.7).

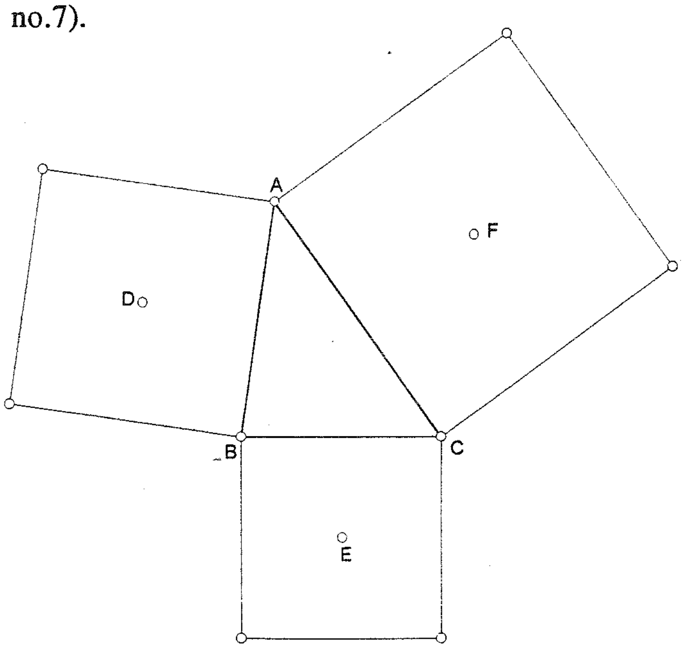


Figure 63

15. (a) Connect the midpoints D, E and F of the squares on the sides of triangle ABC in Figure 63, respectively with C, A and B. What do you notice? Is it always true? Investigate.

(b) Can you generalize?

16. In *Questions and Problems 2*, no.16 we considered a result involving the alternate sides of a kite $2n$ - gon. Formulate a dual for this result and investigate whether it is true or not.

17. (a) Consider the convex cyclic hexagon shown in Figure 64. What do you notice about the sums of the two sets of alternate angles? Is it always true? Investigate.
(b) Can you generalize?
(c) Can you formulate a dual?
(d) What about the converses?

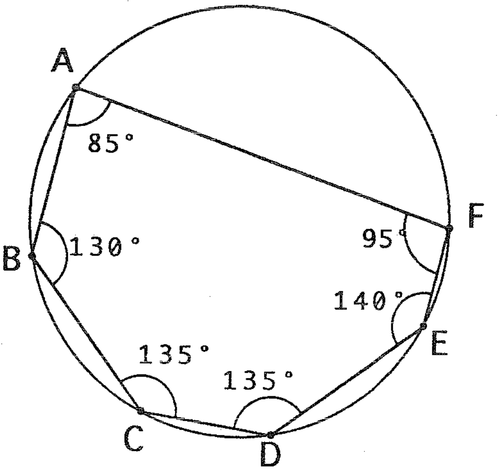


Figure 64

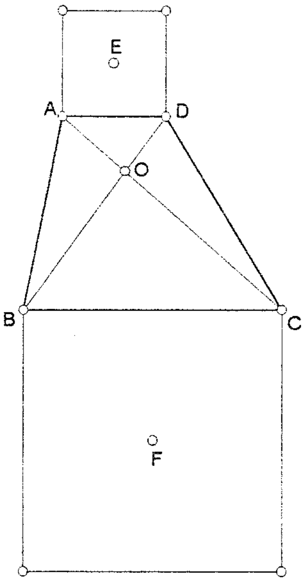


Figure 65

18. (a) Consider the convex trapezium shown in Figure 65 with squares constructed on the two parallel sides. What do you notice about the centres of the squares and the point of intersection of the diagonals? Is it always true? Investigate.
(b) Can you generalize?

19. Are the results shown in Figures 54 and 55 respectively valid for *concave* circum quads and *crossed* cyclic quads? Investigate.

20. In *Questions and Problems 3*, no.9 we considered the construction of squares on the sides of a parallelogram, and that the four centres of these squares themselves form a square.
(a) What happens if we construct squares on the sides of a convex isosceles trapezium and connect the centres as shown in Figure 66? Is it always true? Investigate.
(b) Formulate a dual for your observation in (a) and investigate whether it is true or not.
(c) What happens if we construct squares on the sides of a convex quadrilateral as shown in Figure 67?

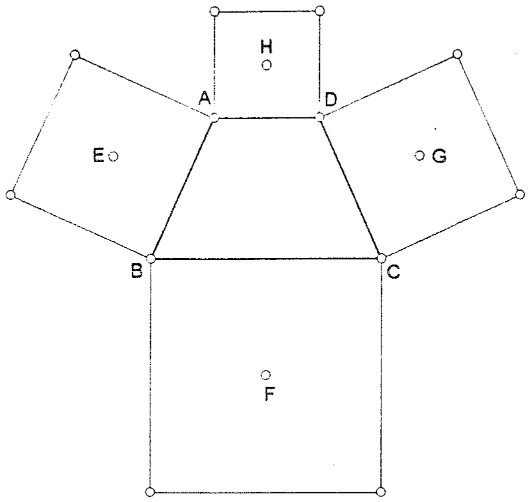


Figure 66

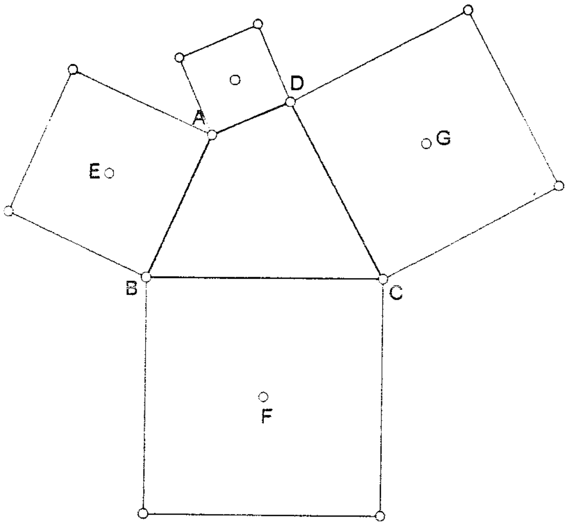


Figure 67

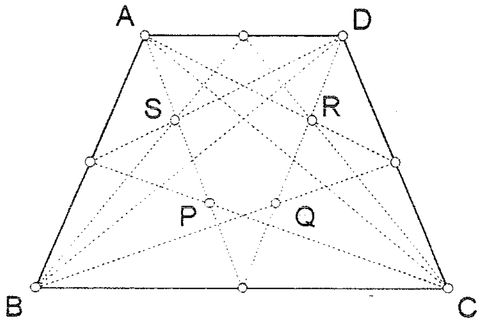


Figure 68

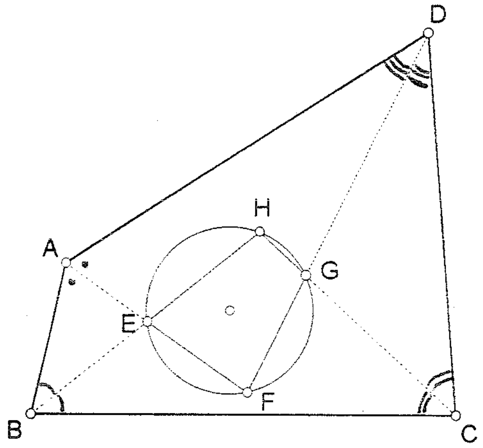
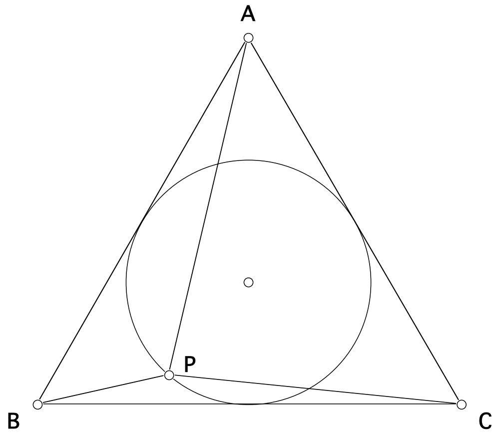


Figure 69

21. (a) Connect the respective centroids P, Q, R and S of triangles ABC, BCD,CDA and DAB for the isosceles trapezium ABCD in Figure 68. What do you notice? Is it always true? Investigate.
(b) Formulate a dual for your observation in (a) and investigate whether it is true or not.

22. (a) Is it a sufficient condition for a quadrilateral to be a circum quad if three of its interior angle bisectors are concurrent? If so, provide an explanation. If not, provide a counter-example.
(b) Formulate a dual to the above question and investigate whether it is true or not.

23. (a) Consider the quadrilateral shown in Figure 69 where the angle bisectors apparently form a cyclic quad EFGH. Is it always true? If so, can you provide an explanation. If not, can you provide a counter-example.
(b) Formulate a dual for Figure 69 and investigate whether it is true or not.



24. (a) Prove that if P is any arbitrary point on the incircle of an equilateral ΔABC, then $PA^2 + PB^2 + PC^2$ is a constant (see above).
    (b) Formulate a dual for the result in (a) and investigate whether it is true or not.

25. (a) Can you generalize the two dual results given in Figures 54 and 55?
    (b) Can you find analogous results related to triangles for the two dual results given for the quadrilaterals in Figures 54 and 55?

26. Can you formulate a dual to the generalized Fermat-Torricelli point discussed in *Solutions 3, No. 1*? If so, investigate whether it is true or not.

Chapter 5

Some generalizations of Pythagoras

There are many generalizations to the Theorem of Pythagoras which can be found by asking appropriate "*what ... if?*" questions such as those mentioned in Chapter 3. Some of these are:

1. The cosine-formula $c^2 = a^2 + b^2 - 2ab\cos C$ for any ΔABC.

2. The theorem of Ptolemy which states that in a convex cyclic quadrilateral ABCD, AB.CD + BC.AD = AC.BD. (If ABCD is a rectangle we have the theorem of Pythagoras).

3. The length of the diagonal d of a n-dimensional rectangular prism with dimensions $x_1; x_2; \ldots x_n$ is determined by

   $$d^2 = \sum_{i=1}^{n} x_i .$$

   (Figure 70 shows a three-dimensional prism where d is determined by $d^2 = x_1^2 + x_2^2 + x_3^2$).

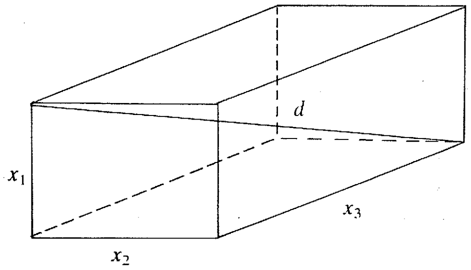


Figure 70

4. The distance d between two points in n-dimensional space with respective coordinates $(x_1; x_2; \ldots x_n)$ and $(y_1; y_2; \ldots y_n)$ is determined by

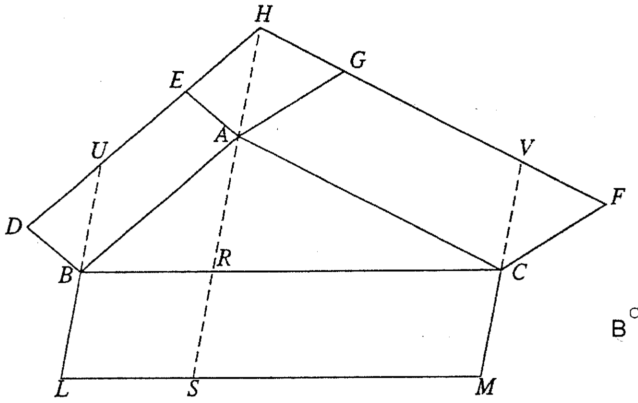
   $$d^2 = \sum_{i=1}^{n} (y_i - x_i).$$

   (Note that generalizations (3) and (4) are equivalent).

5. For any triangle with sides x, y and z such that $z > y \geq x$ or $z < y \leq x$, there exists a number p such that $z^p = x^p + y^p$ (see De Villiers & Du Plooy, 1982).

6. Let ABC be any triangle as shown in Figure 71. Let ABDE and AGFC be any two parallelograms consrtucted externally on AB and AC. Let DE and FG meet in H. Draw BL and CM equal and parallel to HA. Then the area of the parallelogram BCML is equal to the sum of the areas of the two parallelograms ABDE and AGFC. (The special case when the parallelograms are squares and $\angle BAC$ is a right angle

gives us the theorem of Pythagoras. This generalization was given by Pappus (ca. AD. 300) in Book IV of his **Mathematical Collection**).



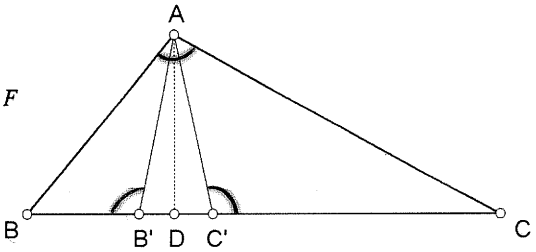
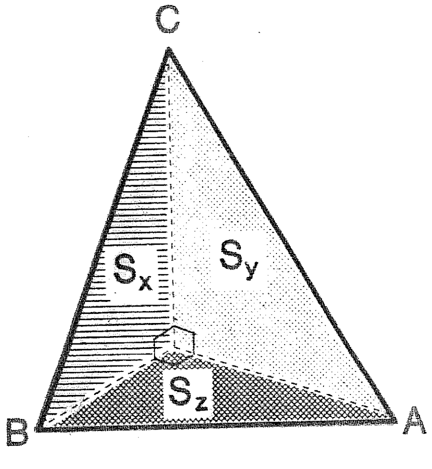


Figure 71 Figure 72

7. Let ABC be any triangle as shown in Figure 72. Construct angles BB'A and CC'A equal to angle BAC. Then $AB^2 + AC^2 = BC(BB' + CC')$. (This result is known as Thabit's generalization).

8. The square of the area S of the plane surface opposite the right angle of a right tetrahedron is equal to the sum of the squares of the areas (S_x, S_y, S_z) of the other plane surfaces (see Figure 73). Or in symbols: $S_x^2 + S_y^2 + S_z^2 = S^2$.

9. If similar figures are constructed on the sides of a right-angled triangle, then the area of the figure on the hypotenuse is equal to the sum of the areas of the other two figures (see Figure 74).



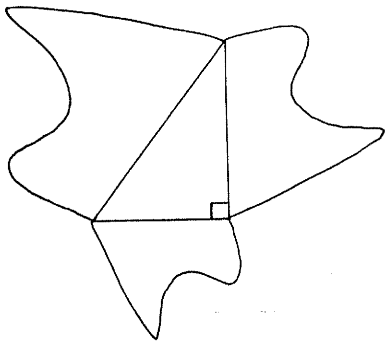


Figure 73 Figure 74

The latter result follows from the easily proved lemma that the areas of similar figures are proportional and therefore that the areas of the similar figures on the sides can respectively be expressed as kc^2, ka^2 and kb^2. But we have that $c^2 = a^2 + b^2$ from which then follows that $kc^2 = ka^2 + kb^2$ by multiplying both sides with k. Therefore the area on the hypotenuse (kc^2) is equal to the sum of the areas on the other two sides $(ka^2 + kb^2)$.

Proofs of the other generalizations are left as exercises to the reader. (Proofs of several of these results can be found in the cover articles of the journal **Pythagoras** from 1991-1994 or in the bibliography at the back). We shall in this chapter only look in more detail at three other generalizations.

Generalizing to *n* dimensions

Since the standard Pythagorean theorem involves the *areas* of squares on the *sides* of a two-dimensional right triangle, one could argue that the analogous result for three dimensions should involve the *volumes* of solids constructed on the *surfaces* of a three-dimensional solid. But what solid would that be?

One way of creating such a solid is the following. Starting from a standard representation of the theorem of Pythagoras, we could translate it a distance of h units in a direction perpendicular to the plane (i.e. into three-dimensional space) to produce a right triangular prism as shown by the second figure in Figure 75. For this solid, the volume of the rectangular prism is clearly equal to the sums of the volumes of the rectangular prisms, A and B, e.g. $c^2h = a^2h + b^2h = h(a^2 + b^2)$.

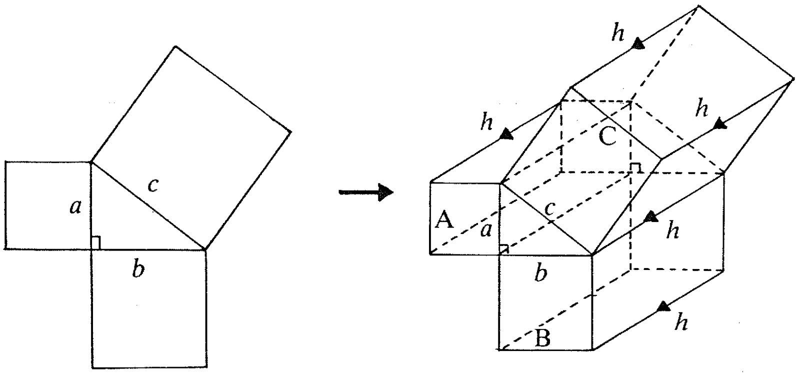


Figure 75

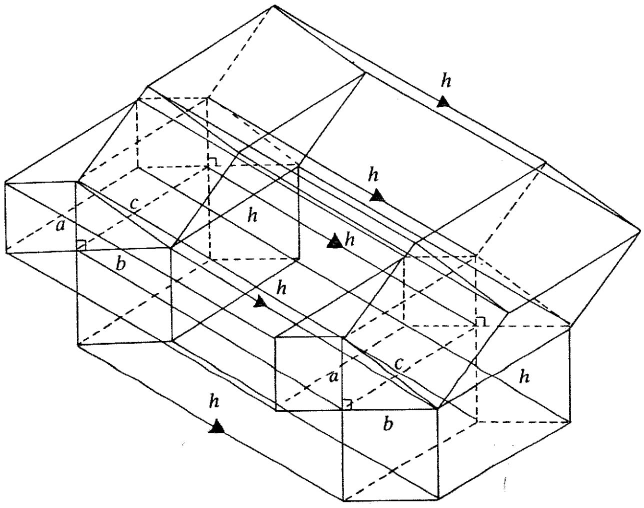


Figure 76

To now further extend the result into four dimensions we need only translate the three-dimensional version a distance of h units within four-dimensional space to obtain a *hyper right triangular prism*. This is shown by the two-dimensional representation in Figure 76. (A *hypercube*, a four-dimensional 'cube', can be formed in the same way by the 'trail' of a cube moving in four-dimensional space). As before we have that the *dimension* (or hyper 'volume') of the hyper rectangular prism formed by C is equal to the sum of the *dimensions* of the hyper rectangular prisms formed by A and B, e.g. $c^2h^2 = a^2h^2 + b^2h^2 = h^2\left(a^2 + b^2\right)$.

If the distances of the translations remain h units we therefore easily have the generalization to n dimensions, namely: $c^2h^{n-2} = h^{n-2}\left(a^2 + b^2\right)$.

On the other hand, if the translations from two- to three-dimensional space and from three- to four-dimensional space are respectively h_1 and h_2 units, we have a more general four-dimensional case: $c^2h_1h_2 = h_1h_2\left(a^2 + b^2\right)$, with its further generalization into n-dimensional space as $c^2h_i = h_i\left(a^2 + b^2\right)$ where $h_i = h_1 \times h_2 \times \ldots \times h_{n-2}$.

Furthermore, if we instead start with the generalization given in (9) we would have the even more general case that $kc^2h_i = kh_i\left(a^2 + b^2\right)$ which can be formulated in words as follows:

"If n-dimensional objects are constructed on the sides of a right-angled triangle with similar figures on its sides by perpendicularly translating it into n-dimensional space, then the dimension $\left(kc^2h_i\right)$ of the n-dimensional object on the hypotenuse is equal to the sum of the dimensions (ka^2h_i and kb^2h_i) of the n-dimensional objects on the other two sides."

Shear similarity

The generalization given in (9) can be further generalized by using the *affine* transformation *shearing* on the similar figures on the sides of a right-angled triangle. In general an affine transformation only preserves the parallelness of corresponding lines, but a shear also preserves *area*.

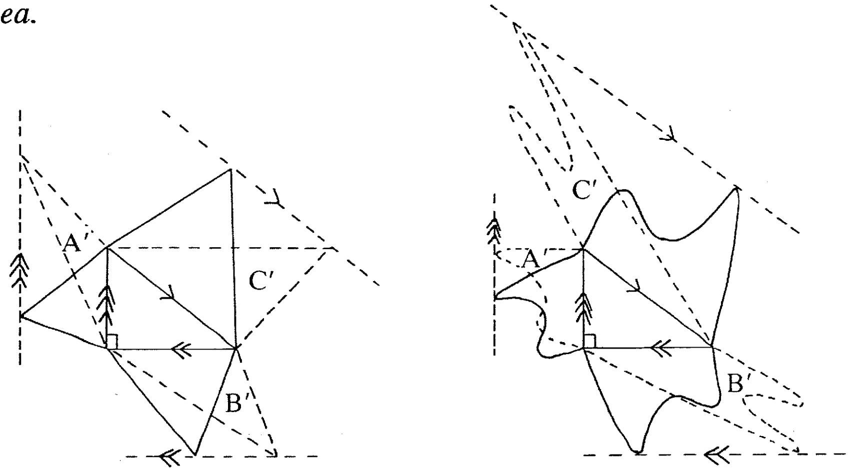


Figure 77

Therefore, by shearing the similar figures on the sides of a right-angled triangle (in directions parallel to the sides), the area of the transformed figure on the hypotenuse would still equal the sum of the areas of the transformed figures on the other sides. For example, the two figures in Figure 77 respectively show the shearing of similar triangles and of similar curvilinear figures on the sides of right-angled triangles. In both cases, we have: area C' = area A' + area B'.

In our previous formulation of the general n-dimensional case above, the plane figures on the sides of the original right-angled triangle need therefore not be similar but merely *shear-similar.* (With shear-similar is meant here that there must exist shears, in directions parallel to the sides of the original right-angled triangle, by which they can be transformed to be similar to each other).

Generalizing to right polygons

"How can we generalize the standard theorem of Pythagoras for right triangles to n-gons in general?"

To answer a question like this, it is often useful to utilize the process of *analogy*. Analogy is a certain kind of similarity: two different objects are analogous if they agree in certain relations of their respective parts. For example, a square and a cube are analogous to each other as both have all sides equal. Similarly, a circle and a sphere are analogous to each other as they can respectively be conceived as the loci in two and three dimensions, equidistant from a point (the centre). Also note that in generalizing to n dimensions earlier, we used an analogy between the concepts area and volume to generalize from two to three dimensions.

This brings us to the question: Which kinds of quadrilaterals, pentagons, hexagons, etc. would be analogous to a right triangle? How can we analogously relate the areas of squares constructed on the sides of such figures to each other?

Let's first look at the quadrilaterals and consider a possible analogue for a right triangle. What about a rectangle which has all its angles equal to 90°? (See Figure 78a). Here the relationship between the areas of the constructed squares can be related to each other as $a^2 + b^2 = a^2 + b^2$, since opposite sides are equal. Therefore for a rectangle, the sum of the squares of two adjacent sides is equal to the sum of the squares of the other two adjacent sides. But is it necessary for it to be rectangle for the result to be true?

In fact, no. We can construct a quadrilateral with two opposite right angles as shown in Figure 78b for which we have the more general result: $a^2 + b^2 = c^2 + d^2$. Clearly this figure is a better analogue of a general right triangle than a rectangle, as it similarly need not have any of its sides equal for the relationship to hold. We will conveniently call it a *right* quadrilateral. (Note that the right triangle on the hypotenuse may lie on the same side as a and b in which case we have a crossed quadrilateral).

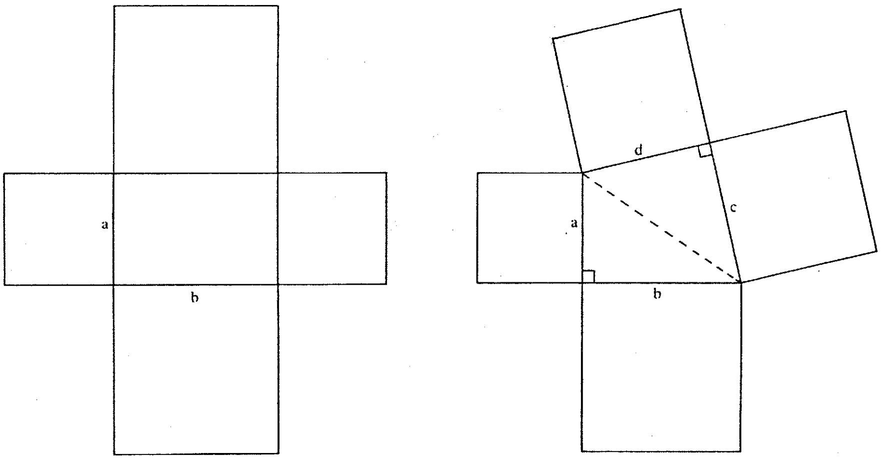


Figure 78

How can we now further extend this result to pentagons, hexagons, etc.?

As a right quadrilateral can be created from a right triangle by merely attaching another right triangle to it, we can expect that we should similarly attach another right triangle to a right quadrilateral in order to obtain a right pentagon. But how should this be done?

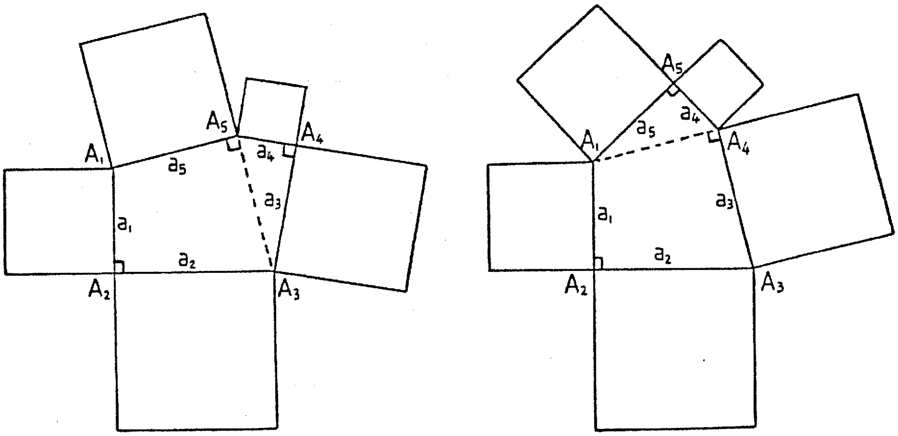


Figure 79

Figure 79 shows how this can be done. For these figures (right pentagons) we now have: $a_1^2 + a_2^2 = a_3^2 + a_4^2 + a_5^2$. By similarly attaching a right triangle to sides a_3, a_4 or a_5 of these pentagons, we can now also create right hexagons, and by repeating the process again, right septagons, and in general, a *right polygon*. We can therefore now generalize the theorem of Pythagoras as follows:

"If a polygon is a right polygon with sides $a_1; a_2; \ldots a_n$, then $a_1^2 + a_2^2 = \sum_{i=3}^{n} a_i^2$."

What about the converse? Is it true or false? Will a polygon with $a_1^2 + a_2^2 = a_3^2 + a_4^2 + \ldots + a_n^2$ necessarily be a right polygon?

Unfortunately not, as a quick quasi-empirical investigation easily leads to the construction of a counter-example such as the one shown in Figure 80, for which $3^2 + 5^2 = 4^2 + (\sqrt{18})^2$ but none of the angles are 90°.

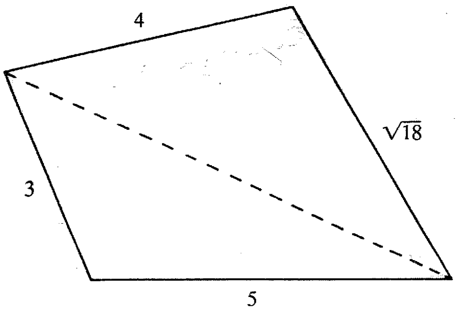


Figure 80

We can even generalize further by also attaching right triangles to sides a_1 and a_2 and renumbering the sides. If we also call such figures right polygons, we would then have: $\sum_{i=1}^{p} a_i^2 = \sum_{i=p+1}^{n} a_i^2$ where n as before is the number of vertices and $2 \le p \le n-1$.

Furthermore, as mentioned in the previous section, the figures constructed on the sides of a right polygon need not be squares nor rectilinear, but need only be shear similar, in which case we have: $\sum_{i=1}^{p} ka_i^2 = \sum_{i=p+1}^{n} ka_i^2$.

This result can also be further extended into m-dimensional space by translating any right polygon with shear similar figures on its sides by distances $h_1, h_2, \dots h_{m-2}$ perpendicularly into higher dimensions. Algebraically it can be expressed as: $\sum_{j=1}^{p} ka_j^2 h_i = \sum_{j=p+1}^{n} ka_j^2 h_i$.

Some further generalizations of these results are given in De Villiers (1991, 1992a &b).

Chapter 6

Generalizing Varignon's theorem

One of the most fascinating aspects of mathematics, as already pointed out in Chapter 3, is that it is an evolutionary process in the sense that solutions to problems often lead to further problems, provided one retains an enquiring and open mind to continue asking questions. As Daltry (1966:20) put it: *"..the end of problem-solving may not be solutions so much as new problems."* A particularly useful problem posing strategy is to always consider the possibility of generalizing a particular result, no matter how trivial it may seem, or even when a conjectured generalization turns out to be false.

What follows is an illustrative example of the application of this strategy to Varignon's theorem as illustrated in Figure 81, namely, that if the midpoints E, F, G and H of the adjacent sides of any quadrilateral ABCD are consecutively connected, then EFGH is a parallelogram (also see *Questions and Problems 3,* no. 5). According to Coxeter & Greitzer (1967:53), the first known published proof of this rather simple result was only given in 1731 by Pierre Varignon, and the inscribed parallelogram is consequently often referred to as the Varignon parallelogram. It should again be pointed out that the investigations which follow in this chapter are particularly suited for exploration or demonstration on dynamic software like *Cabri Géomètre* or *Geometer's Sketchpad*.

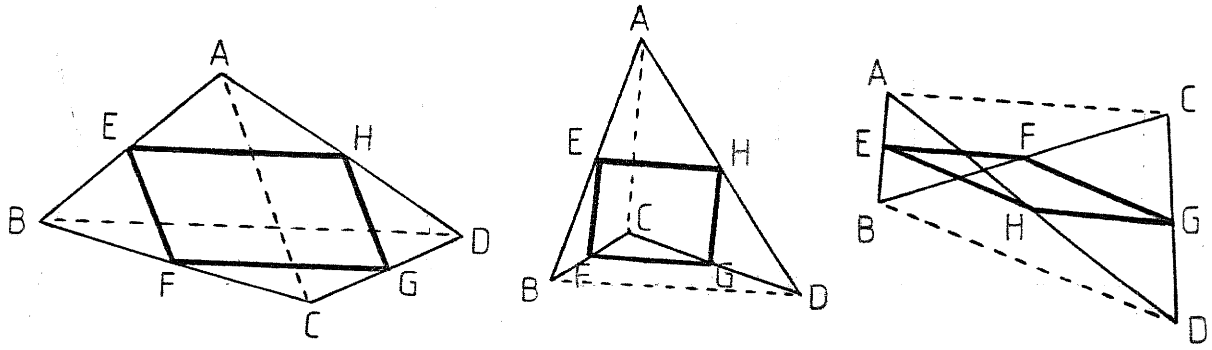


Figure 81

How can we generalize this result? What about generalizing it to other polygons? Is it really necessary that E, F, G and H are midpoints of the sides? Or phrased differently, how can we maintain the result if E, F, G and H are not midpoints?

A first generalization

Let us first look at the generalization to other polygons. Since the result deals with a quadrilateral which has an even number of vertices, we skip the pentagons and firstly consider the possibility of an analogous result for hexagons. If we consecutively connect the midpoints of the adjacent sides of a regular hexagon, we find another regular hexagon as shown in Figure 82a. In other words, we obtain a hexagon with opposite sides parallel and equal, i.e. a *parallelo-hexagon.* (Also see Solutions 2, no. 17). Of course, this is not true in general, for if we repeat the construction with the hexagon shown in Figure 82b, we do not find a hexagon with opposite sides parallel and equal. The question now arises: under which conditions would we find a hexagon with opposite sides parallel and equal?

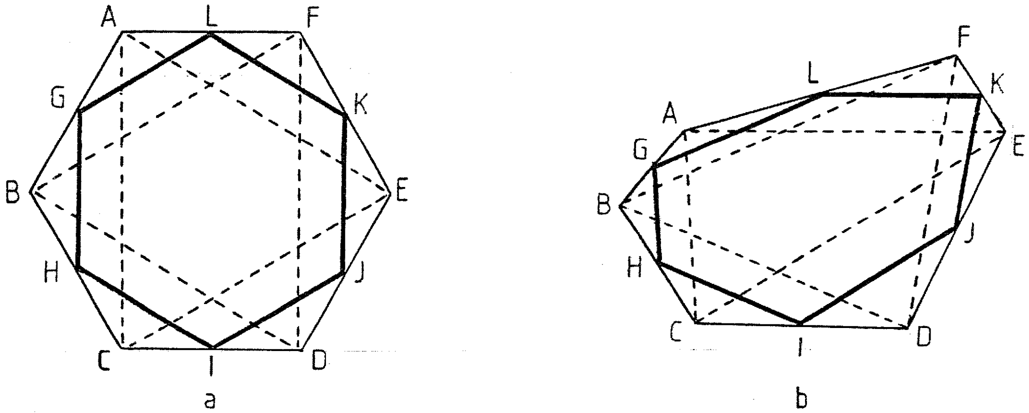


Figure 82

Well, let's just for a moment reflect back on the deductive explanation for the quadrilaterals and the characteristic property upon which it depends (Figure 81). By drawing a diagonal (or both), we simply utilize the theorem that the line segment connecting the midpoints of two sides of a triangle is parallel and equal to half the length of the third side. Now bearing this explanation in mind, and looking at the hexagon in Figure 82b, it should immediately be clear that GH would be parallel and equal to the opposite side JK if AC //= DF. Similarly, it follows that BD //= EA ⇒ HI //= KL and CE //= FB ⇒ IJ //= LG. In other words, in any hexagon with the aforementioned properties we would find an inscribed parallelo-hexagon.

What is more, a hexagon ABCDEF with AC //= DF, BD //= EA and CE //= FB, would itself be a parallelo-hexagon. For example, AC //= DF ⇒ ACDF is a parm ⇒ AF //= CD. The same is true for the other two pairs of opposite sides. Three examples of parallelo-hexagons with the midpoints of their sides consecutively connected to form inscribed parallelo-hexagons are shown in Figure 83. Note that the parallelo-hexagons ABCDEF can be drawn by constructing two congruent triangles ACE and DFB with AC//DF, AE//DB and CE//FB, and then connecting A, B, C, D, E and F. (Also compare the three inscribed parallelo-hexagons. What relationship is noticeable between them? Why?)

By using the same reasoning, we can now easily extend the result to an octagon ABCDEFGH with the properties that AC //= EG, BD //= FH, CE //= GA and DF //= HB. As shown in Figure 84, inscribed parallelo-octagons are formed if we consecutively connect the midpoints of the sides of such figures. Note however, that in this case the octagons ABCDEFGH are not necessarily parallelo-octagons. Such octagons can easily be drawn by constructing any two parallelograms ACEG and BDFH and then simply connecting A, B, C, D, E, F, G and H.

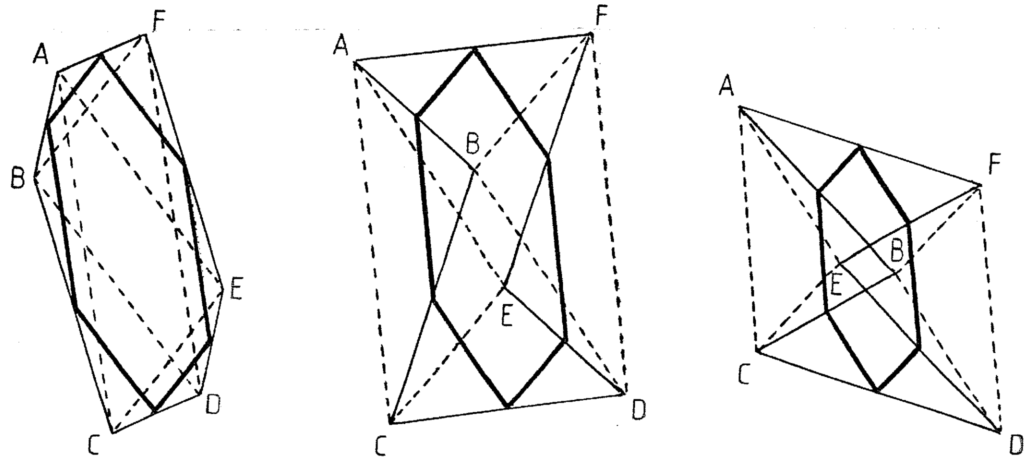


Figure 83

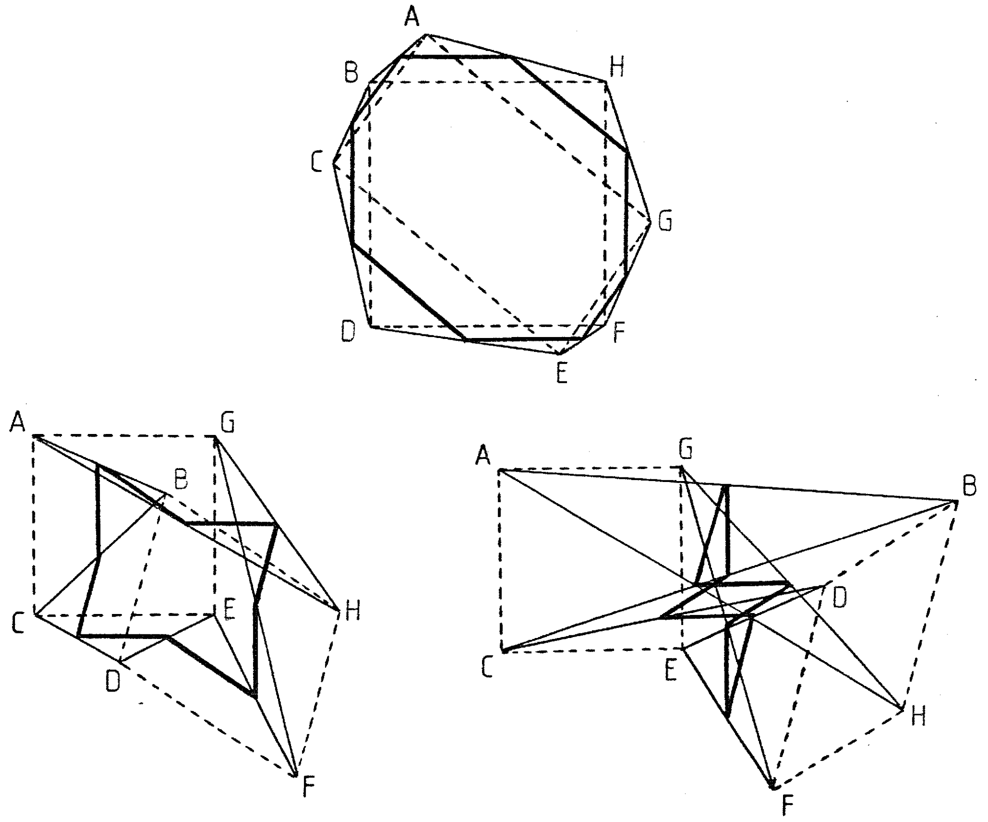


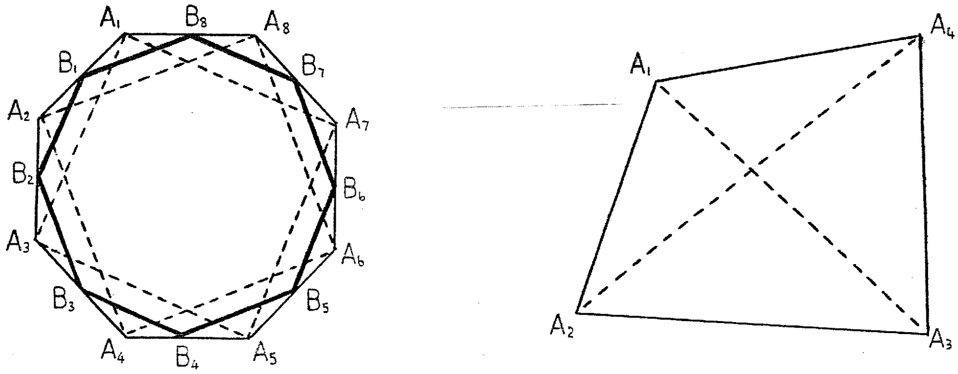
Figure 84

We can now generalize the result as follows:

(1) "If $A_1A_2...A_{2n}$ ($n > 1$) is any $2n$-gon with A_iA_{i+2} //= $A_{i+n}A_{i+n+2}$ ($i = 1; 2; ...n$) and B_j are the midpoints of A_jA_{j+1} ($j = 1; 2; ...2n$), then $B_1B_2...B_{2n}$ is a parallelo-$2n$-gon".

Proof

The proof is straight forward, following directly from the theorem that the line segment connecting the midpoints of two sides of a triangle is parallel and equal to half the length of the third side. For example, as illustrated in Figure 85a where $n = 4$, B_jB_{j+1} //= $\frac{1}{2}A_iA_{i+2}$ and $B_{j+n}B_{j+n+1} = \frac{1}{2}A_{i+n}A_{i+n+2}$. But it is given that A_iA_{i+2} //= $A_{i+n}A_{i+n+2}$ which implies that B_jB_{j+1} //= $B_{j+n}B_{j+n+1}$ and therefore that $B_1B_2 \ldots B_{2n}$ is a parallelo-2n-gon.



a b

Figure 85

This result is of course further generalizable to 2n-gons with the property A_iA_{i+2}//$A_{i+n}A_{i+n+2}$ or $A_iA_{i+n+2} = A_{i+n}A_{i+n+2}$ for which the respective inscribed 2n-gons would have B_jB_{j+1}// $B_{j+n}B_{j+n+1}$ or $B_jB_{j+1} = B_{j+n}B_{j+n+1}$. (Note that a 2$n$-gon with opposite sides parallel necessarily implies opposite sides equal, and vice versa, only in the case of a parallelogram). Let's also briefly look at the special case of the quadrilaterals (Figure 85b). According to the condition of the above theorem, we should in the case of a quadrilateral $A_1A_2A_3A_4$ have A_1A_3 //= A_3A_1 and A_2A_4 //= A_4A_2, which means that each diagonal must be parallel and equal to itself. But this is obviously true for any orientation or lengths of the diagonals and accounts for the variation of quadrilaterals shown in Figure 81.

The general theorem (1) also has the following interesting corollary, the proof of which is left to the reader:

(2) "The perimeter of the inscribed parallelo-2n-gon is equal to the sum of A_iA_{i+2} (or to half the sum of A_jA_{j+2} where $j = 1; 2; \ldots 2n$)".

In the special case of the quadrilaterals, this simply means that the perimeter of the inscribed quadrilateral $B_1B_2B_3B_4$ is equal to the sum of the diagonals of the quadrilateral $A_1A_2A_3A_4$ (compare Denson, 1989).

A second generalization

Let us now return to the original result and critically examine the necessity of B_1, B_2, B_3 and B_4 being midpoints of the sides (see Figure 86). As mentioned before, the result depends on B_1B_2 // A_1A_3 // B_3A_4 and B_1B_4 // A_2A_4 // B_2A_3. How can we maintain these relationships if B_1, B_2, B_3 and B_4 are not midpoints of the sides?

Well, the line segment B_1B_4 would remain parallel to A_2A_4 provided the two sides A_1A_2 and

A_1A_4 are divided in the same proportion (ratio) by B_1 and B_4. Since the same is true for the other line segments, $B_1B_2B_3B_4$ would clearly be a parallelogram if

$$\frac{A_1B_1}{B_1A_2}=\frac{A_1B_4}{B_4A_4}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}.$$

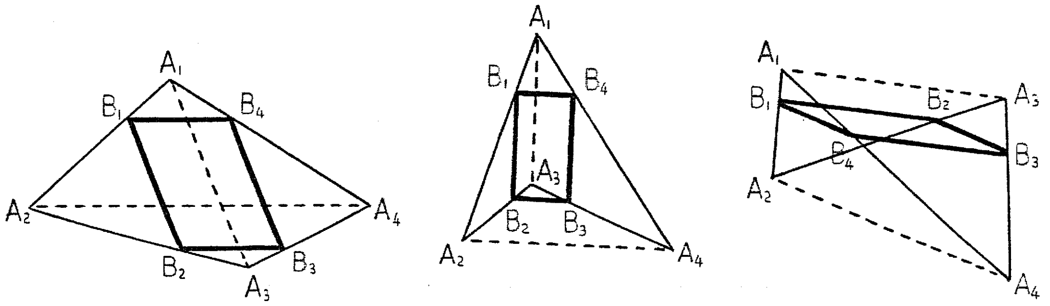


Figure 86

Spontaneously our next questions arise:

(i) What is the relationship between the perimeter of such an inscribed parallelogram and the diagonals of the original quadrilateral? Has it changed or does it stay the same?

(ii) Is this result also generalizable to the $2n$-gons described before?

With regard to the first question, we unfortunately find that the perimeter is no longer necessarily equal to the sum of the diagonals. In fact, if we let

$$\frac{A_1B_1}{B_1A_2}=\frac{A_1B_4}{B_4A_4}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}=\frac{p}{q}$$

then it can be easily shown that the perimeter is equal to $2(qA_1A_3+pA_2A_4)/(p+q)$. Interestingly, if $A_1A_3=A_2A_4$, this formula reduces to $2A_1A_3$ or $2A_2A_4$.

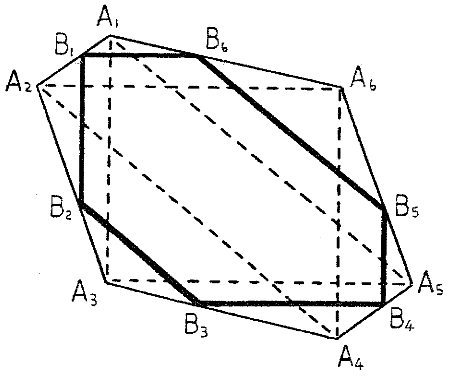


Figure 87

Let us now consider the second question above in relation to the parallelo-hexagon $A_1A_2A_3A_4A_5A_6$ shown in Figure 87. Careful investigation shows that if

$$\frac{A_1B_1}{B_1A_2}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}=\frac{A_5B_4}{B_4A_4}=\frac{A_5B_5}{B_5A_6}=\frac{A_1B_6}{B_6A_6}$$

then $B_1B_2B_3B_4B_5B_6$ would clearly be a hexagon with opposite sides parallel. (Note: In Solutions 2, no.17 (Figure 2.31) we called such a hexagon a *parallel-hexagon*, and its generalization to polygons, a *parallel-2n-gon*). In fact, A_iA_{i+2} need not be parallel and equal to $A_{i+3}A_{i+5}$ for this particular result to be true, but only parallel. We can therefore now formulate the following generalization:

(3) "If $A_1A_2...A_{2n}$ ($n > 1$) is any $2n$-gon with $A_iA_{i+2} // A_{i+n}A_{i+n+2}$ ($i = 1; 2;...n$) and B_j are points on A_jA_{j+1} ($j = 1; 2;... 2n$) so that for $k = 1; 3; 5;...2n - 1$:

$$\frac{A_kB_k}{B_kA_{k+1}} = \frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}} = \frac{p}{q},$$

then $B_1B_2...B_{2n}$ is a parallel-$2n$-gon (opposite sides parallel)."

Proof

The proof follows directly from the theorem that if a line segment connecting two points divides two sides of a triangle into equal ratios then it is parallel to the third side. For example, it implies that $B_jB_{j+1}//A_iA_{i+2}$ and $B_{j+n}B_{j+1+n}//A_{i+n}A_{i+n+2}$. But since $A_iA_{i+2}//A_{i+n}A_{i+n+2}$ we have $B_jB_{j+1}//B_{j+n}B_{j+n+1}$ and therefore that $B_1B_2...B_{2n}$ is a parallel-$2n$-gon.

Of course, in the special case of a quadrilateral we obtain a parallelo-$2n$-gon (parallelogram), since opposite sides parallel implies opposite sides equal. Furthermore, we have the following corollary of theorem (3), proof of which is also left to the reader:

(4) "The perimeter of the inscribed parallel-$2n$-gon is equal to

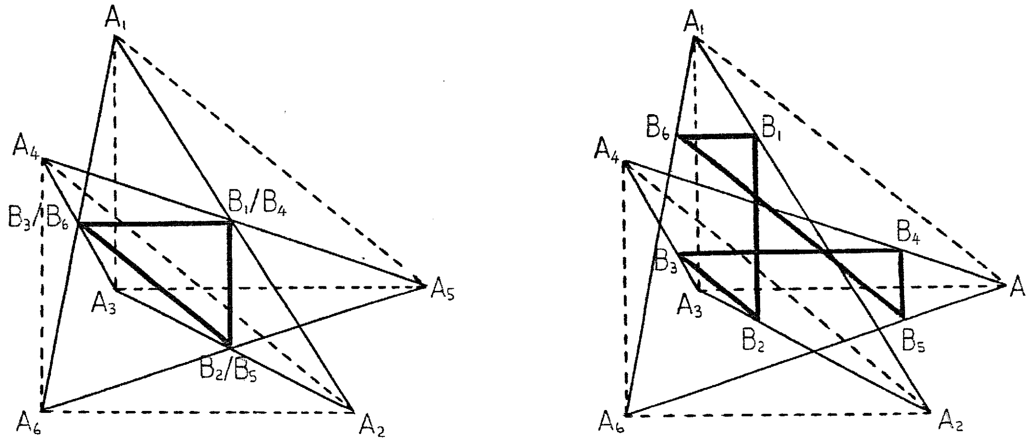
$$\frac{q\sum_{i=1}^{n} A_{2i-1}A_{2i+1} + p\sum_{i=1}^{n} A_{2i}A_{2i+2}}{p+q}."$$

If we have $\sum A_{2i-1}A_{2i+1} = \sum A_{2i}A_{2i+2}$ the perimeter reduces to $\sum A_{2i-1}A_{2i+1}$ or $\sum A_{2i}A_{2i+2}$. Of course, if A_jA_{j+2} are all of equal length, let's say d, then the perimeter simply becomes nd.

A counter-example

As mentioned in Chapter 3, quasi-empirical testing/experimentation often plays an important part in the development of a piece of new mathematics. However, it is possibly one of the most neglected aspects when it comes to the teaching of mathematics. Quasi-empirical testing/experimentation is useful since it not only gives us confidence in the validity of our conjectures and theorems, but also an essential concrete understanding/appreciation of their meaning and domains of validity which is sometimes not revealed by logical deduction. More importantly, it can produce counter-examples which necessitate either abandonment or reformulation of conjectures, definitions and/or proofs.

Earlier it was mentioned and proved that *any* hexagon with $A_iA_{i+2} //= A_{i+3}A_{i+5}$ is a parallelo-hexagon. Let's test this observation by trying to construct a hexagon with this property, but different from those shown in Figures 82a, 83 and 87. In the last two cases, the hexagons were drawn by constructing two congruent triangles ACE ($A_1A_3A_5$)and DFB ($A_4A_6A_2$)with AC//DF ($A_1A_3//A_4A_6$), AE//DB ($A_1A_5//A_4A_2$) and CE//FB ($A_3A_5//A_6A_2$), and then connecting A(A_1), B(A_2), C(A_3), D(A_4), E(A_5) and F(A_6). Or in other words, by first constructing ΔACE ($A_1A_3A_5$) and then rotating it through 180° to map onto ΔDFB($A_4A_6A_2$). However, we can also maintain the parallelness of A_iA_{i+2} and $A_{i+3}A_{i+5}$ by translating $\Delta A_1A_3A_5$ to map onto $\Delta A_4A_6A_2$, and then connecting the vertices to obtain the figures shown in Figure 88.



a

b

Figure 88

These figures are clearly not parallelo-hexagons as opposite sides are not parallel and equal, but in fact are unequal in length and intersect. In this case, the earlier given proof is false since it is based on the implicit assumption that the opposite sides A_iA_{i+1} and $A_{i+3}A_{i+4}$ do not intersect, and is therefore applicable only to the type of configuration shown in Figures 82a, 83 and 87. However, this example does not invalidate generalizations (1) and (3), as illustrated respectively by Figures 88a and 88b. Also note that in the first case opposite sides B_iB_{i+1} and $B_{i+3}B_{i+4}$ actually coincide to produce a degenerate parallelo-hexagon in the form of a triangle. (This happens because the opposite sides of $A_1A_2 ... A_6$ intersect in their midpoints).

Considering converses

Let's now consider possible converses of Theorems 1 and 3. In both these theorems we connected points which divided the sides of a $2n$ -gon proportionally, to obtain another $2n$ -gon with opposite sides equal and parallel, or just parallel. Conversely, we should therefore now consider the case where we start at the midpoints, or any other point B_1 of a particular side, say A_1A_2 and drawing $B_1B_2//A_1A_3$, $B_2B_3//A_2A_4$, etc. with B_2, B_3, etc. on A_2A_3, A_3A_4, etc.

For example, what happens if we start with the midpoint B_1 of side A_1A_2 of the hexagon with $A_iA_{i+2}//= A_{i+3}A_{i+5}$ shown in Figure 89a, and we draw $B_jB_{j+1}//A_jA_{j+2}$? Similarly, what happens if we start out with any point B_1 of side A_1A_2 of the hexagon with

$A_iA_{i+2}//=A_{i+3}A_{i+5}$ shown in Figure 89b, and we draw $B_jB_{j+1}//A_jA_{j+2}$? What happens if A_iA_{i+2} is only parallel to $A_{i+3}A_{i+5}$, but not equal? What happens if we choose B_1 on any of the other sides?

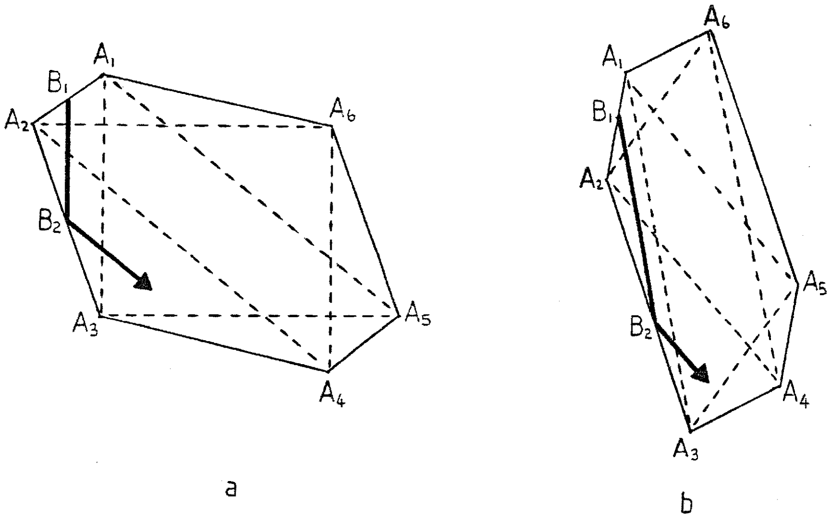


Figure 89

Complete the figures above by continuing to draw $B_jB_{j+1}//A_jA_{j+2}$. What do you notice? Also construct hexagons with A_iA_{i+2} only parallel to $A_{i+3}A_{i+5}$, but not equal, and repeat the previous exercise with B_1 a midpoint or any other point. What do you notice? Can you formulate and explain your observations?

Although intuitively one may anticipate that one could go on indefinitely without necessarily returning to the original starting point, we find the rather surprising result that we always return to the start no matter where the starting points are chosen. Furthermore, in the two figures above we always find a parallelo-hexagon, while in the second case with A_iA_{i+2} only parallel to $A_{i+3}A_{i+5}$, always a parallel-hexagon. These observations can now be generalized to the following corresponding converses to Theorems 1 and 3:

(5) "If $A_1A_2 \dots A_{2n}(n>1)$ is any $2n$ - gon with $A_iA_{i+2}//= A_{i+n}A_{i+n+2}$ $(i = 1; 2;\dots n)$ and B_jB_{j+1} is drawn parallel to A_jA_{j+2} $(j = 1;\ 2;\dots 2n)$ starting from the midpoint B_1 of A_jA_{j+1}, then a *closed* parallelo-$2n$-gon $B_1B_2\dots B_{2n}$ is formed"

(6) "If $A_1A_2\dots A_{2n}(n>1)$ is any $2n$-gon with $A_iA_{i+2}//A_{i+n}A_{i+n+2}$ $(i = 1; 2;\dots n)$ and B_jB_{j+1} is drawn parallel to A_jA_{j+2} $(j = 1; 2;\dots 2n)$ starting from any point B_1 on a side A_jA_{j+1}, then a *closed* parallel-$2n$ - gon $B_1B_2\dots B_{2n}$ is formed."

Proof

We shall now only prove that the 2n-gons referred to above are closed, leaving the rest of the two proofs to the reader to complete. Consider the case where B_1 is on A_1A_2. It is therefore required for us to prove that the point B_{2n+1} is the same as B_1. Since B_jB_{j+1} is drawn parallel to A_jA_{j+2} we have as before:

$$\frac{A_1B_1}{B_1A_2}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}=\frac{A_5B_4}{B_4A_4}=\ldots \text{ etc., or in general:}$$

$$\frac{A_kB_k}{B_kA_{k+1}}=\frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}}=\frac{p}{q} \quad \text{where } k = 1;\ 3;\ 5;\ldots$$

Considering $k = 2n + 1$ we will therefore have that

$$\frac{A_1B_1}{B_1A_2}=\frac{A_3B_2}{B_2A_2}=\ldots=\frac{A_{2n+1}B_{2n+1}}{B_{2n+1}A_{2n+2}}=\frac{A_{2n+3}B_{2n+2}}{B_{2n+2}A_{2n+2}}.$$

Compare the first and third ratios above. Since A_{2n+1} and A_{2n+2} are the same points as A_1 and A_2 respectively, we therefore have that the point B_{2n+1} and B_1 are the same point. Similarly, if we choose B_1 on any of the other sides, we can show that B_1 and B_{2n+1} are identical.

Looking back

However, looking back and reflecting on our preceding proof, it should be clear that we did not at all utilize the property $A_iA_{i+2}//A_{i+n}A_{i+n+2}$. This proof therefore immediately provides us with the following interesting generalization (illustrated for a hexagon in Figure 90):

(7) "If $A_1A_2 \ldots A_2(n > 1)$ is any $2n$-gon and B_jB_{j+1} is drawn parallel to A_jA_{j+2} (j = 1; 2;...$2n$) starting from any point B_1 on a side A_jA_{j+1}, then a *closed* $2n$-gon $B_1B_2 \ldots B_{2n}$ is formed".

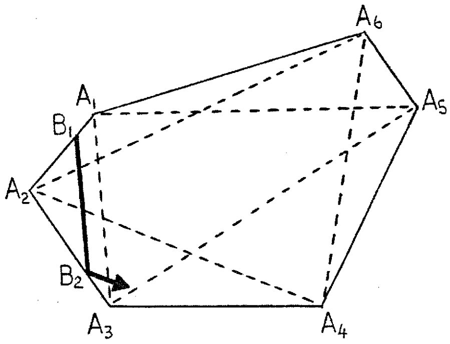


Figure 90

(It is left to the reader to complete the figure in Figure 90).

An analogous result

Let's now examine what happens if we carry out the same procedure of drawing $B_jB_{j+1}//A_jA_{j+2}$ on a triangle, pentagon, septagon, etc. The reader is now invited to carry out this procedure on the two figures shown in Figure 91. What do you notice? Can you form a generalization?

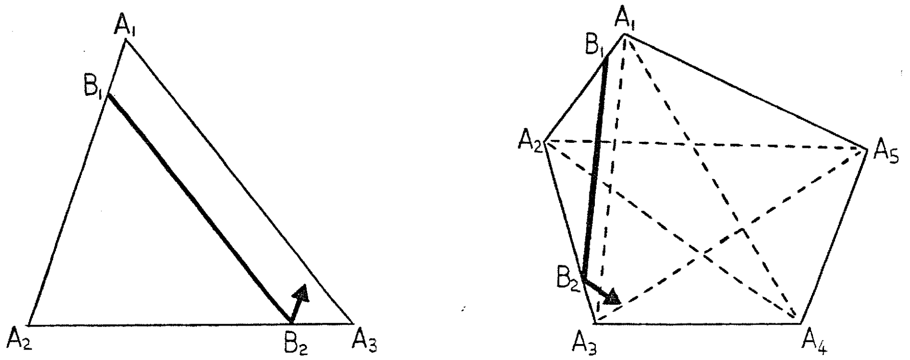


Figure 91

Investigating some cases the following generalization analogous to Theorem 7 can be made:

(8) "If $A_1A_2...A_{2n-1}$($n>1$) is any ($2n$-1)-gon and B_jB_{j+1} is drawn parallel to A_jA_{j+2}(j = 1; 2;...$4n$-2) starting from any point B_1 on a side A_jA_{j+1}, then a *closed* ($4n$-2)-gon $B_1B_2...B_{4n-2}$ is formed".

Proof

The proof is straight forward, for example, consider the case where B_1 is on A_1A_2. Let us now first consider the point B_{2n} which will also be on A_1A_2. Since B_jB_{j+1} is drawn parallel to A_jA_{j+2} we have as before:

$$\frac{A_kB_k}{B_kA_{k+1}} = \frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}} = \frac{p}{q} \text{ where } k = 1; 2; 3; 5;...$$

Considering $k = 2n - 1$ we will have that :

$$\frac{A_1B_1}{B_1A_2} = ... = \frac{A_{2n-1}B_{2n-1}}{B_{2n-1}A_{2n}} = \frac{A_{2n+1}B_{2n}}{B_{2n}A_{2n}}.$$

Compare the first and third ratios above. Since A_{2n+1} and A_{2n} are the same points as A_2 and A_1 respectively, it clearly follows that B_1 and B_{2n} will be the same point only when B_1 is the midpoint.

Let us now show that B_{4n-1} is the same point as B_1. Considering $k = 4n - 1$ we will have that:

$$\frac{A_1B_1}{B_1A_2} = ... = \frac{A_{4n-1}B_{4n-1}}{B_{4n-1}A_{4n}} = \frac{A_{4n+1}B_{4n}}{B_{4n}A_{4n}}.$$

Compare the first and second ratios above. Since A_{4n-1} and A_{4n} are the same points as A_1 and A_2 respectively, we therefore have that the point B_{4n-1} and B_1 are the same point. Similarly, if we choose B_1 on any of the other sides, we can show that B_1 and B_{4n-1} are identical.

Some reflections

The generalizations described in this chapter would probably not have been possible to make simply by blind, trial and error experimentation, since an understanding of the *explanatory* property which makes it true for quadrilaterals, proved indispensable throughout the whole exploration. In particular, the generalization from Theorems (5) and (6) to Theorem 7 is a good example of the *discovery* function of proof described in Chapter 3, whereby the deductive indentification of the explanatory property of a particular result often enables further generalization.

It is furthermore important to distinguish between two different kinds of generalization demonstrated here, namely *inductive* generalization and *deductive* generalization. With inductive generalization is meant here that a generalization is initially made on quasi-empirical grounds without necessarily any deductive thought involved, for example observing and formulating generalizations like Theorems 5 and 6 from the consideration of some particular cases like the figures in Figure 9. A deductive generalization on the other hand is made on the basis of a logical deduction, for example by deductively analysing the conditions of a particular theorem (or theorems) and finding from its proof that a specific condition is sufficient, but not necessary, thereby enabling further generalization. The generalization of Theorem 7 from the proofs of Theorems 5 and 6 is therefore also a good example of deductive generalization.

Further questions

In keeping with the spirit of this book, inquisitive readers may wish to follow up with questions like the following for further exploration, or add questions of their own:

(a) Can you find analogous results of the preceding theorems involving equal-$2n$-gons? (Opposite sides equal - see Solutions 2, no.17).

(b) (i) Can you draw a quadrilateral configuration to give a degenerate parallelogram when the midpoints of the sides are connected? What figure can be anticipated?
(ii) Can you draw an octagon configuration to give a degenerate parallelo-octagon when the midpoints of the sides are connected? What figure can be anticipated?

(c) Specialization is often also a useful problem posing strategy. For example, consider the following two special cases of the original theorem (see Figure 53):
(i) If the midpoints of the sides of any perpendicular quad are connected, then we obtain a rectangle. Can you generalize this result?
(ii) If the midpoints of the sides of any diagonal quad are connected, then we obtain a rhombus. Can you generalize this result?
(iii) Further specialization would be to ask under what conditions would we obtain a square and to try and generalize it.

(d) A further interesting property of the original theorem for convex quadrilaterals is that the area of the inscribed parallelogram EFGH is half the area of the quadrilateral ABCD (see Figure 81). (Hint: compare the area of ΔAEH with that of ΔABD, the area of ΔBFE with that of ΔBCA, etc.).

(i) Is this result true for concave and crossed quadrilaterals?

(ii) Can you generalize it?

(e) Consider the case shown in Figure 92 where the midpoints of the sides of a triangle have been connected. Does it have any properties which are generalizable to (2*n*-1)-gons? Can you extend the result to other points on the sides of a triangle and generalize to (2*n*-1)-gons?

(f) Consider the regular hexagon (star of David) shown in Figure 93. If the midpoints of the sides are connected as shown, another star of David is obtained. Can you generalize this result?

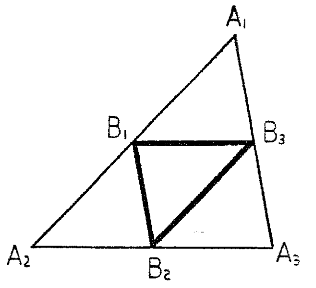


Figure 92

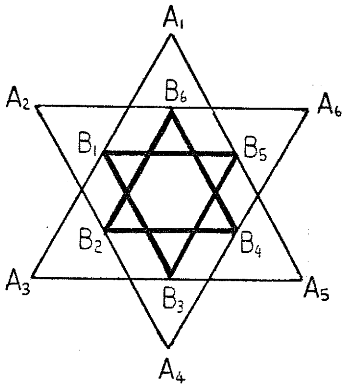


Figure 93

(g) Another result directly related to the original theorem is given in Coxeter & Greitzer (1967:54), namely: the segments B_1B_3 and B_2B_4 joining the midpoints of the opposite sides of a quadrilateral and the segment C_1C_2 joining the midpoints of the diagonals are concurrent and bisect one another (see Figure 94). Can you also generalize this result?

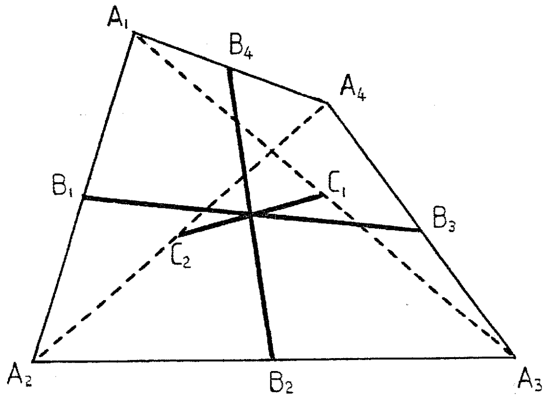


Figure 94

(h) We can specialize the original theorem in another way. For example, if sides A_1A_2 and A_2A_3 are lying in a straight line as shown in Figure 95 and the midpoints are connected as before, an inscribed parallelogram is obtained in the triangle $A_1A_3A_4$. We can therefore inscribe a parallelogram in any triangle by consecutively connecting the midpoints of two sides with the midpoints of any two subdividing sections of the third side (in which case the one side of the parallelogram coincides with part of the third side and is half its length). Can you generalize this result?

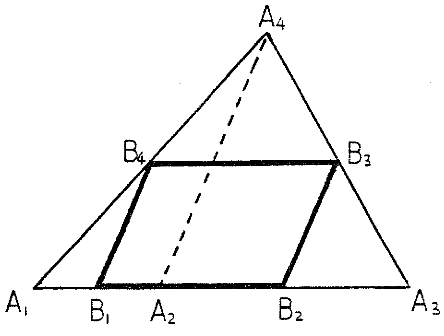


Figure 95

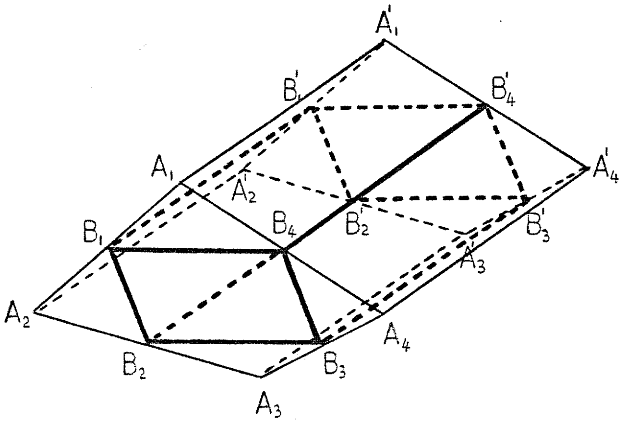


Figure 96

(i) Suppose we translate a quadrilateral $A_1A_2A_3A_4$ with an inscribed Varignon parallelogram $B_1B_2B_3B_4$ some distance into three dimensional space to obtain their images $A'_1A'_2A'_3A'_4$ and $B'_1B'_2B'_3B'_4$, and consider the three dimensional figures traced out by the translation of the vertices. As shown in Figure 96 we would then have a *parallelo-piped* (a solid bounded by six parallelograms as its faces, the opposite pairs being congruent and parallel) inscribed in a *prism* (a solid with two parallel congruent polygons as opposite faces with edges joining corresponding vertices so that the remaining faces are parallelograms). Can you generalize this result as before? Can you apply the theorems in this chapter, as well as the area relationship mentioned in question (d) above? Can you further generalize to n dimensions?

Chapter 7

Generalizing some geometrical gems

In Ross Honsberger's excellent book **Mathematical Gems III** the reader is introduced to the so-called "*equilic quadrilateral*", namely a quadrilateral ABCD with one pair of opposite sides equal, say AD = BC, which are inclined at 60° to each other. (The latter condition might also be stated in the form $\angle A + \angle B = 120^\circ$). Then the following engaging results are presented and proved. (Also see *Questions & Problems 3,* no's. 11-14).

1. If ABCD is an equilic quadrilateral, then the midpoints P, Q and R of the diagonals and the side CD always determine an equilateral triangle (see Figure 97).

2. If an equilateral triangle PCD is drawn outwardly on CD, then triangle PAB is also equilateral (see Figure 98).

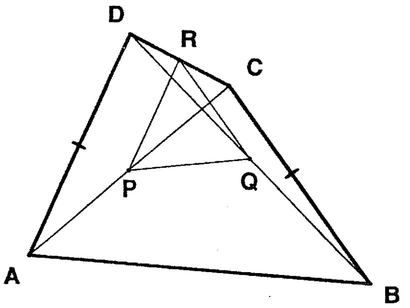


Figure 97

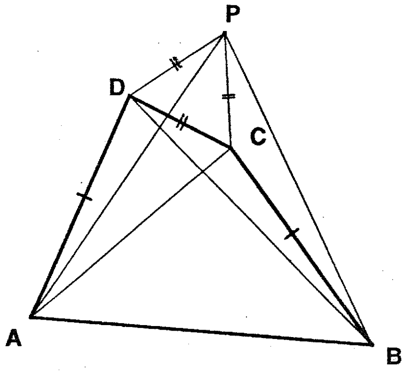


Figure 98

3. If equilateral triangles are drawn on AC, DC and DB, away from AB, then the three new vertices, P, Q and R are collinear (see Figure 99).

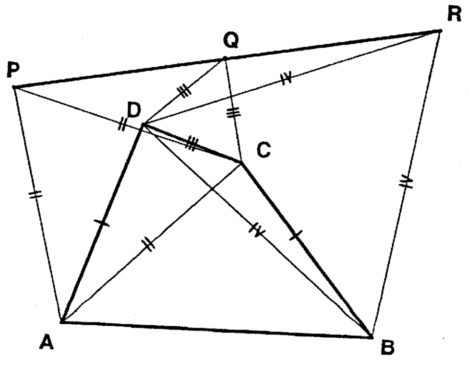


Figure 99

Before proceeding further, first try and prove these results and consider possible generalizations (if not already done so earlier).

Some "what-if" questions

Questions like the following "*what-if*" questions seem appropriate to ask and investigate with regard to the preceding three problems:

- Under which conditions might triangles PQR and PAB, respectively in the first and second results, be *isosceles* triangles?
- What happens if ABCD is *concave* or *crossed*? Are the results still valid?
- What happens if, in all three cases, ABCD is any quadrilateral with opposite sides equal? (In other words, in general, if ABCD is a *side quad* - see Solutions 2, no. 17 or Solutions 4, no. 3).
- What happens if triangle PCD in the second result is *isosceles*?
- What happens if triangles PAC, QDC and RDB are *similar* to each other in the third result? Would P, Q and R still be collinear?

The following generalizations were the direct result of asking and investigating questions like those above.

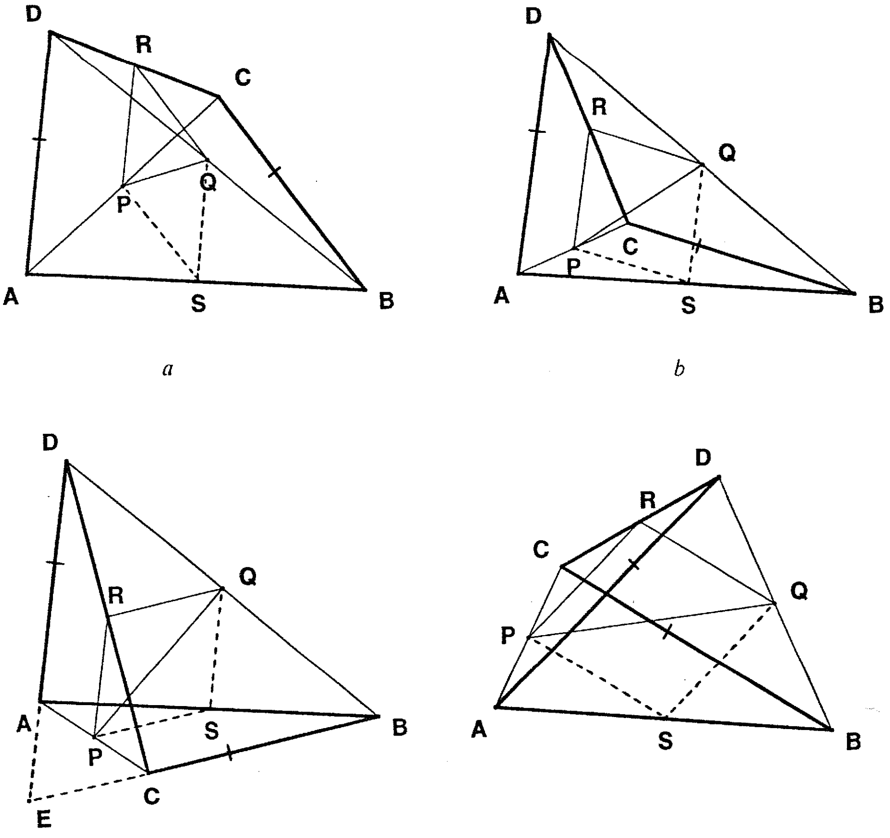
Generalizing the first result

Basically, the first result can be proved by first showing that PR = QR since AD = BC (e.g. PR = ½AD and QR = ½BC, but AD = BC ⇒ PR = QR), and then that $\angle PRQ = 60^\circ$ since PR and QR are inclined towards each other at the same angle as AD and BC. Therefore triangle PQR is equilateral. (An isosceles triangle with an angle of 60° is equilateral).

Looking back
If we drop the second condition that AD and BC are inclined toward each other at 60°, the following interesting generalization is immediately apparent from the above proof:

4. If ABCD is a "*side quad*" (a quadrilateral with AD = BC), then the midpoints P, Q and R of the diagonals and the side CD always determine an isosceles triangle with PR = QR and $\angle PRQ = 180^\circ - \angle A - \angle B$ (see Figure 100).

In addition, if we connect P and Q with the midpoint S of side AB, then it easily follows that PRQS is always a rhombus (eg. PS //= QR and SQ //= PR). From the latter argument we also immediately have the further generalization that, in any quadrilateral, PRQS will always be a parallelogram. Note however that when ABCD is convex (Figure 100a) or crossed as shown in Figure 100c, AD could be parallel to BD, in which case the respective parallelograms PQRS simply degenerate into straight lines (e.g. in the first case P and Q coincide, and in the second case R and S).



a *b*

c *d*

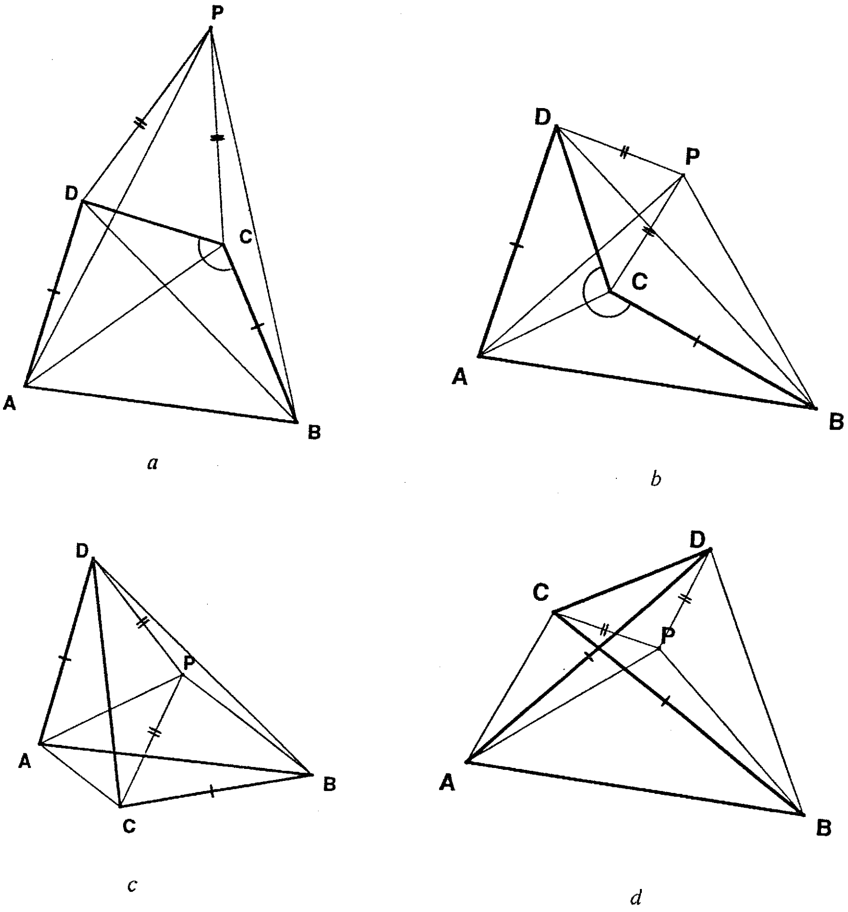
Figure 100

Furthermore, in Figures 100a, 100b and 100d it is easy to see that PR and QR are always inclined towards each other at the same angle as AD and BC, and therefore that $\angle PRQ = 180° - \angle A - \angle B$. In Figure 100c however we need to consider $\angle ABC$ as negative in relation to $\angle DAB$ for this relationship to hold. For example, if we continually decrease $\angle B$ in Figure 100b by rotating side BC anticlockwise, it will eventually become zero when coinciding with AB, and then negative in Figure 100c. Suppose we now let $\angle DAB = \angle A$ and $\angle B = -\angle ABC$, and extend DA and BC to meet in E. Then in triangle AEB we have $\angle EAB =$ $180° - \angle A$ and $\angle AEB = 180° - (180° - \angle A + \angle ABC) = \angle A - \angle ABC = \angle A + \angle B$. Since PR and PS are inclined towards each other at the same angle as DA and BC, we have $\angle RPS = \angle A + \angle B$. But since RQ // PS, $\angle PRQ$ and $\angle RPS$ are supplementary and we have that $\angle PRQ = 180° - \angle A - \angle B$.

Generalizing the second result

Basically, the second result can be proved by first showing that $\angle ADP = \angle BCP$. This implies that triangles ADP and BCP are congruent $(s, \angle, s)$, from which follows that AP = BP. The desired result can then be obtained by simply showing that $\angle APB = 60°$. Further investigation shows that it is possible to maintain the congruency between triangles ADP and BCP, and therefore the equality of AP and BP, more generally as follows:

5. If an isosceles triangle PCD is drawn on side CD of a side quad ABCD so that $\angle PDC = \angle PCD = \frac{1}{2}(\angle A + \angle B)$, then triangle PAB is an isosceles triangle with PA = PB (see Figure 101).



a *b* *c* *d*

Figure 101

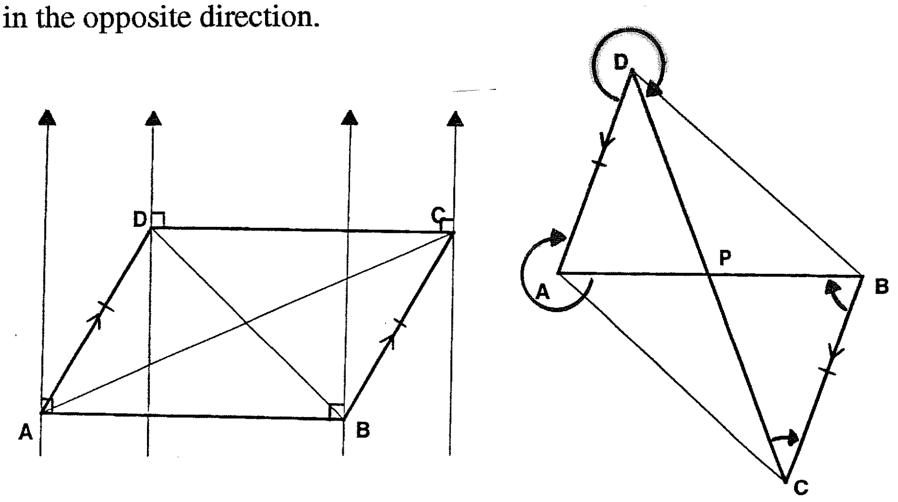
The following proof covers the convex and concave cases shown in Figures 101a and 101b. The proofs for the other two configurations are similar, requiring only minor modifications, and are left as exercises to the reader.

Proof

Considering all the angles around the point C, we have $\angle BCP = 360° - (\angle BCD + \angle PCD)$. But in quadrilateral ABCD we have $\angle BCD = 360° - \angle A - \angle B - \angle ADC$ and $\angle PCD = \frac{1}{2}(\angle A + \angle B)$ from the construction, and substituting these into the first equation we obtain $\angle BCP = \frac{1}{2}(\angle A + \angle B) + \angle ADC = \angle PDC + \angle ADC = \angle ADP$. Thus triangles ADP and BCP are congruent $(s, \angle, s)$ and therefore PA = PB.

In addition, in triangle DPC we have $\angle APC = 180° - \angle PDC - \angle PCD - \angle DPA = 180° - \angle A - \angle B - \angle DPA$. Since $\angle APB = \angle APC + \angle CPB = 180° - \angle A - \angle B - \angle DPA + \angle CPB$, and $\angle DPA = \angle CPB$ from the above congruency, we therefore have $\angle APB = 180° - \angle A - \angle B$. But since triangle PAB is isosceles, it follows that $\angle PAB = \angle PBA = \frac{1}{2}(\angle A + \angle B)$ and therefore triangle PAB is also similar to triangle PDC.

It should furthermore be noted that when $\angle ABC$ becomes greater than $\angle DAB$ in Figure 101c, then $\angle PDC$ and $\angle PCD$ both become negative and triangle DPC is constructed inward, in other words, in the opposite direction.



a *b*

Figure 102

This result can also be conceptualized as follows in the special cases when AD//BC in the configurations shown in Figures 101a and 101c. Since $\angle A + \angle B = 180°$ in the first case, we can construct perpendicular lines on CD through C and D as shown in Figure 102a. Since the co-interior angles are supplementary, these two lines are also parallel. However, in the projective plane parallel lines meet at infinity, i.e. at the so-called *vanishing point.* Let the vanishing point be P', then from symmetry we have P'D = P'C. However, perpendicular lines on AB through A and B are also parallel to the aforementioned parallels, thus also meeting at the same point P' as before, and therefore P'A = P'B.

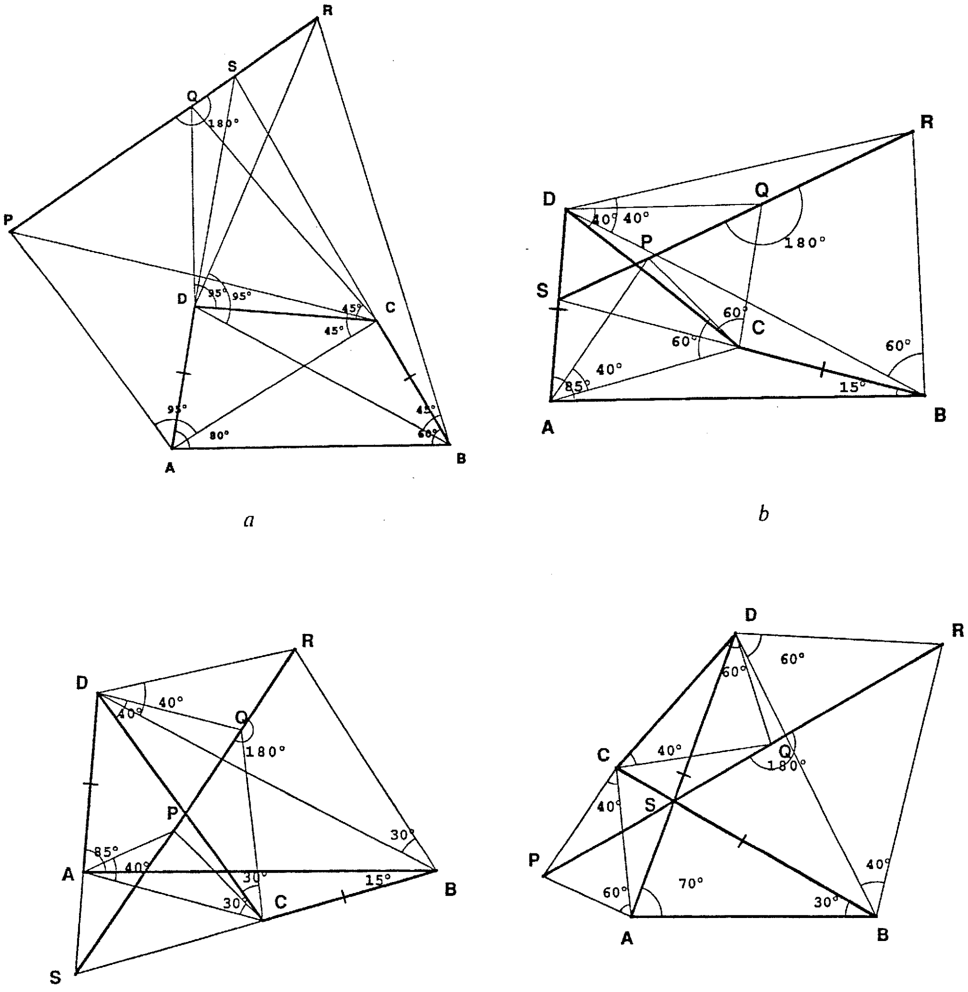
In the second case $\angle A + \angle B = 360°(= 0°)$, since ABCD is a crossed quad. Therefore vertex P of the isosceles triangle PCD would be the midpoint of CD as shown in Figure 102b, and therefore from the property of parallelograms we have PA = PB.

Generalizing the third result

From the preceding generalization, the author first conjectured the generalization that P, Q and R would remain collinear if similar isosceles triangles PAC, QDC and RDB with $\angle PAC = \angle PCA = \frac{1}{2}(\angle A + \angle B)$ were constructed on AC, DC and DB of any side quad. Although accurate construction and measurement with *Cabri Geometre* quickly verified this conjecture, additional investigation lead to the following further generalization:

6. If similar triangles PAC, QDC and RDB are constructed on AC, DC and DB of any side quad ABCD so that $\angle PAC + \angle PCA = \angle A + \angle B$, then P, Q and R are collinear (see Figure 103).

(Although the above generalization may seem to suggest a further generalization of the fifth result, namely if triangle PCD is drawn so that $\angle PDC + \angle PCD = \angle A + \angle B$, then triangle PAB would be similar to triangle PCD, this unfortunately turns out false as can easily be verified by construction and measurement).



a b

c d

Figure 103

The condition that $\angle PAC + \angle PCA = \angle A + \angle B$ may also be more conveniently stated as $\angle APC = 180° - \angle A - \angle B$ or $\angle APC = \angle ASB$. In addition, if we extend AD and BC to meet in S, we find that S is collinear with P, Q and R. Examples of four different cases are shown in Figure 103.

The following proof is with reference to Figure 104 and covers the convex and concave cases, but the proofs for the other two cases are similar, requiring only minor modifications, and are left as exercises to the reader. After several unsuccessful attempts at proving this result, the author then noticed, while manipulating the configuration with *Sketchpad,* that $\angle ACB = \angle APQ$, thereby enabling him to formulate the following proof. (This is therefore another good example of how quasi-empirical investigation, as mentioned in Chapter 3, can assist in the construction of a proof). Basically the proof involves first showing $\angle ACB = \angle APQ$ and $\angle ADB = \angle QRB$, and then that PQ and QR have the same direction.

Construction

Connect P with Q and Q with R. Construct CE //= DA as shown. Call the point F where CE cuts AB.

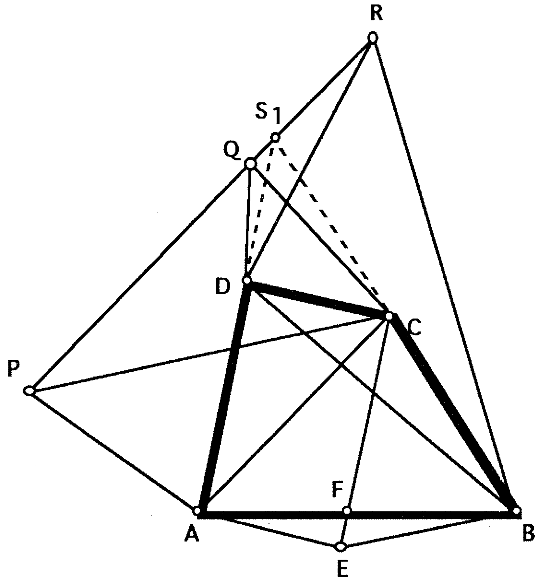


Figure 104

Proof

Therefore ADCE is a parallelogram and $\angle CAE = \angle DCA$ (alternate) and $\angle CFB = \angle A$ (corresponding). In triangle CFB we therefore have $\angle ECB = 180° - \angle A - \angle B = \angle APC$.

From the similarity of triangles PAC and QDC, we have $\angle PCA = \angle QCD$. Therefore $\angle PCA + \angle PCD = \angle QCD + \angle PCD$ which implies that $\angle DCA = \angle QCP$, and therefore $\angle CAE = \angle QCP$ ... (1). From the similarity between triangles PAC and QDC we also have:

$$\frac{PC}{AC} = \frac{QC}{DC}.$$

But DC = AE since ADCE is a parallelogram, and therefore:

$$\frac{PC}{AC} = \frac{QC}{AE} \text{ ... (2).}$$

Thus, according to (1) and (2) above, triangles AEC and CQP are similar. Therefore $\angle ACE = \angle CPQ$. Since $\angle APC = \angle ECB$ as already shown, we have $\angle ACE + \angle ECB = \angle APC + \angle CPQ$ and therefore $\angle ACB = \angle APQ$. By constructing DG //= CB we can similarly prove that $\angle ADB = \angle QRB$.

A counterclockwise rotation of size $\angle PAC$ around A carries C into AP, say C', and B to B'. But since $\angle ACB = \angle APQ$ we have $\angle AC'B = \angle APQ$ and therefore C'B'//PQ. Thus PQ is inclined to BC at an angle of size $\angle PAC$. Similarly, from a clockwise rotation of size $\angle DBR$ around B, we have that A'D'//QR, that is, QR is inclined to AD at an angle of $-\angle DBR$.

Since AD and BC are inclined towards each other at $180° - \angle A - \angle B = 180° - \angle PAC - \angle DBR$, rotating BC through $\angle PAC$ and AD through $-\angle DBR$ (in appropriate directions), aligns B'C' and A'D' up in the same direction, and we therefore have that PQ and QR also line up in the same direction, i.e. P, Q and R are collinear.

Further construct $\angle QS_1D = \angle QCD$ with S_1 on QR. Connect S_1 with C. We shall now prove $\angle DS_1C = 180° - \angle A - \angle B$ and that S_1CB and S_1DA are straight lines, and therefore that S_1 and S are the same point.

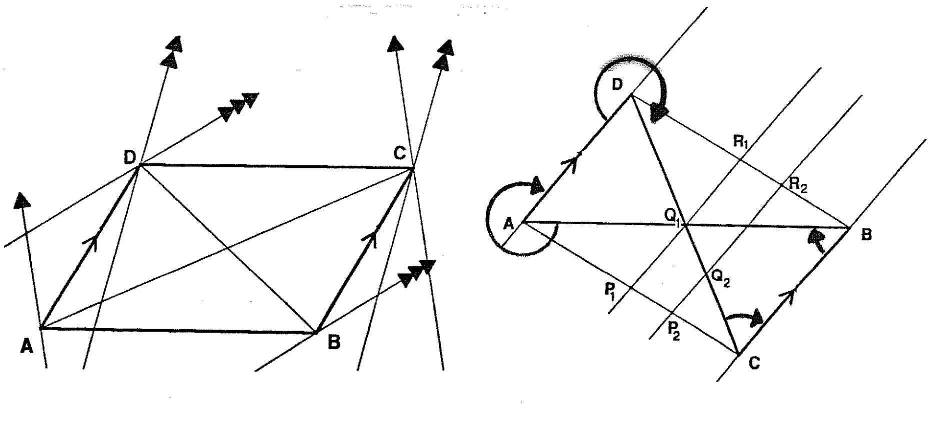
From the construction we have $QDCS_1$ a cyclic quadrilateral (equal angles on same chord). Thus $\angle DS_1C = \angle DQC = 180° - \angle A - \angle B$ on chord DC. Also $\angle S_1CD = \angle PQD$.

In triangle PQC, we have $\angle PQD = 180° - \angle CPQ - \angle QCP - \angle DQC = \angle S_1CD$. But $\angle BCD = \angle DCA + \angle ACE + \angle ECB$ and $\angle DCA = \angle QCP$ (proved earlier), $\angle ACE = \angle CPQ$ (proved earlier) and $\angle ECB = \angle DQC$ (construction). Therefore $\angle S_1CD + \angle BCD = 180°$ and S_1CB is a straight line. Similarly we can prove that S_1DA is a straight line. Therefore S_1 and S are the same point, namely the intersection of AD and BC. (Note: if S falls on QP we simply construct $\angle QS_1C = \angle QDC$ and show in the same manner that S_1 and S are the same point).

Looking back

Carefully looking back at the above proof, the author realized that he had never used the property that AD = BC; in other words, that the result was immediately generalizable to *any* quadrilateral! This again illustrates the indispensable value of an explanatory proof which enables one to generalize a result by the identification of the fundamental properties upon which it depends.

It is possible, as before, to also conceptualize this result in the special case when AD//BC in convex and crossed configurations as follows. (Note: ABCD need not be a parallelogram).



a *b*

Figure 105

Since $\angle A + \angle B = 180°$ in the first case, we can construct three sets of parallel lines inclined at the same angles on respectively AC, DC and DB as shown in Figure 105a. These three sets of parallel lines respectively meet at the vanishing points P', Q' and R', but in the projective plane these vanishing points all lie on the ideal line, i.e. are collinear. Furthermore, AD and BC also meet in the vanishing point S' which is collinear with the aforementioned points.

Since $\angle A + \angle B = 360°(= 0°)$ in the second case, the points P, Q and R respectively lie on AC, DC and DB. If we therefore choose P, Q and R so that AP/PC = DQ/QC = DR/RB (in order to maintain the similarity of the degenerate triangles) then it follows easily that P, Q and R would be collinear as shown by the two examples in Figure 105b. In addition, since the sides are divided in ratio, we have that the line through P, Q and R is parallel to sides AD and BC, and therefore meet in the same vanishing point S'. Thus, S', P, Q and R are collinear.

It should perhaps also be pointed out that when sides AD and BC intersect as shown in Figure 103d, it is necessary to construct triangle QDC "*inward*". This construction can easily be conceptualized by continually decreasing AB in Figure 103b, and keeping AD and BC constant, until they intersect. Note also, when $\angle ABC$ becomes greater than $\angle DAC$ in Figure 103c, that triangles PAC, QDC and RDB must be constructed in the opposite direction.

A dual for the first result and its generalization

Due to the duality between *side* quads and *angle* quads (quadrilaterals with at least one pair of opposite angles equal) as discussed in Chapter 4, it is interesting to consider whether there are corresponding duals regarding the preceding three generalizations. For example, with regard to the first generalization (Result 4), we may conjecture the following dual:

7. If ABCD is an angle quad with $\angle B = \angle D$, then the angle bisectors of $\angle AYB$, $\angle DXA$ and $\angle A$ always determine an isosceles triangle (see Figure 106).

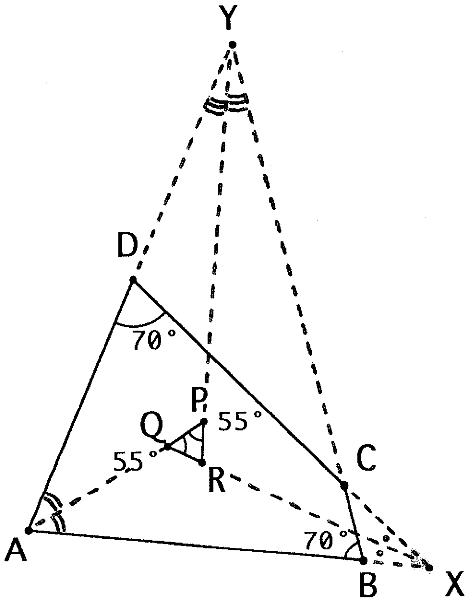


Figure 106

(Note: Here the *angle bisector* of $\angle A$ (or $\angle C$) is taken as the dual of the *midpoint* of side CD (or AB), and the *angle bisectors* of the *angles* between *opposite sides* ($\angle AYB$ and $\angle DXA$) as the duals of the *midpoints* of the *diagonals* connecting *opposite vertices/angles*).

Interestingly, this conjecture turns out to be true as shown in Figure 106, and can easily be proved in general as follows.

Proof

In ΔAYB, we have $\angle AYB = 180° - \angle A - \angle B$ so that $\angle AYP = 90° - \frac{1}{2}\angle A - \frac{1}{2}\angle B$. Therefore $\angle QPR = \angle YAP + \angle AYP = 90° - \frac{1}{2}\angle B$. In a similar way from ΔDXA, it follows that $\angle PQR = 90° - \frac{1}{2}\angle D$. But $\angle B = \angle D$ is given, and therefore $\angle QPR = \angle PQR$. Furthermore, it then follows that $\angle PRQ = \angle B = \angle D$ and that ΔPQR would be equilateral when $\angle B = \angle D = 60°$.

However, it does not seem possible to formulate corresponding duals for angle quads for the second and third results and their generalizations. The problem is to formulate meaningful duals corresponding to the construction of equilateral or isosceles triangles on the sides and diagonals of side quads.

Further questions

Can you prove the following results from Honsberger (1985) related to equilic quadrilaterals ABCD with AD = BC and $\angle A + \angle B = 120°$? Can you further generalize any of these results? Investigate.

1. Equilateral triangles drawn outwardly on AD, AB and BC yield three new vertices which determine an equilateral triangle.
2. Equilateral triangles drawn outwardly on AD and DC, and inwardly on BC, yield three new vertices that determine an equilateral triangle.
3. Equilateral triangles drawn inwardly on AD, CD and BC yield three new vertices which determine an equilateral triangle.
4. Reflect ABCD in AB to get hexagon ADCBC'D'. Then equilateral triangles drawn outwardly on either set of alternate sides yield three new vertices which determine an equilateral triangle.

Chapter 8

Dual generalizations of Von Aubel's theorem

In *Questions and Problems 3*, no. 9 we considered the result that the centres of squares on the sides of a parallelogram form a square, and in *Questions and Problems 4*, no. 20, its generalization, Von Aubel's theorem, namely, that the centres of squares on the sides of any quadrilateral, form a quadrilateral with equal and perpendicular diagonals.

How could we further generalize these results? What happens if instead of squares on the sides we construct similar rectangles or rhombi?

Some years ago when the author first came across Von Aubel's theorem in Martin Gardner's book **Mathematical Circus** (1981:176-179), he intuitively sensed that it might be generalizable to similar rectangles and rhombi. After some initial experimentation on the arrangement of the similar rectangles and rhombi, it was found to be indeed possible as follows.

Let us first consider the special case when the base quadrilateral is a parallelogram. If we construct similar rectangles on the sides of a parallelogram as shown in Figure 107, we find EFGH a *rhombus* ; in other words, the diagonals EG and FH are still *perpendicular*, but not necessarily equal any more.

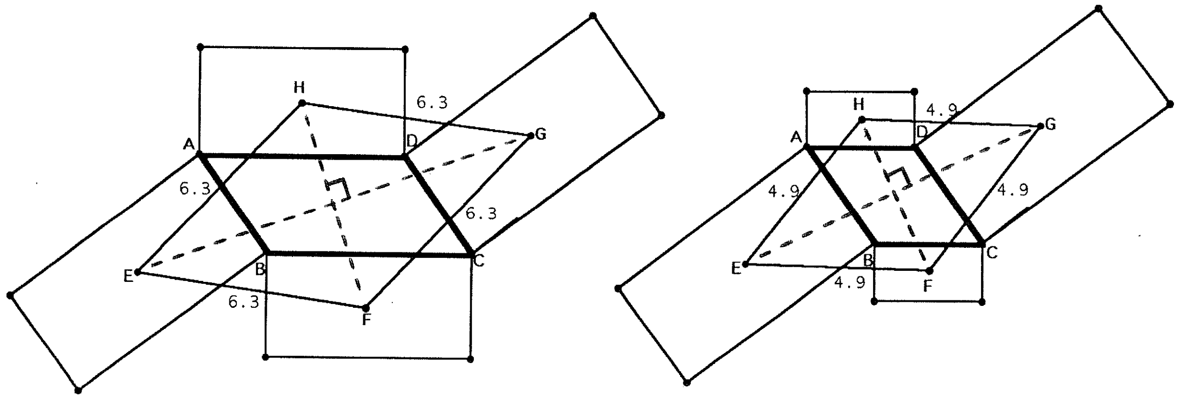


Figure 107

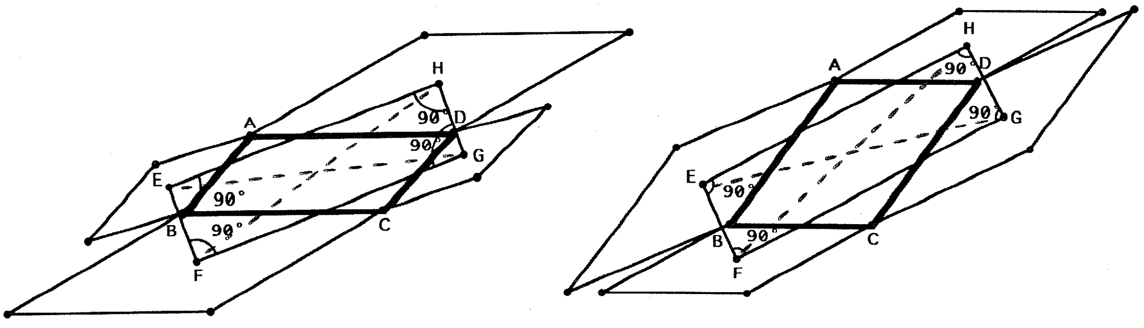


Figure 108

Similarly, we find the dual result that if similar rhombi are constructed on the sides of a parallelogram as shown in Figure 108, then EFGH is a *rectangle*; in other words, the diagonals are still *equal,* but not necessarily perpendicular any more.

These two results now immediately suggest the following dual generalizations:

1. If similar *rectangles* are constructed on the sides of any quadrilateral as shown in Figure 109, then the centres of these rectangles form a *perpendicular quad* (a quadrilateral with perpendicular diagonals)

2. If similar *rhombi* are constructed on the sides of any quadrilateral as shown in Figure 110, then the centres of these rhombi form a *diagonal quad* (a quadrilateral with equal diagonals).

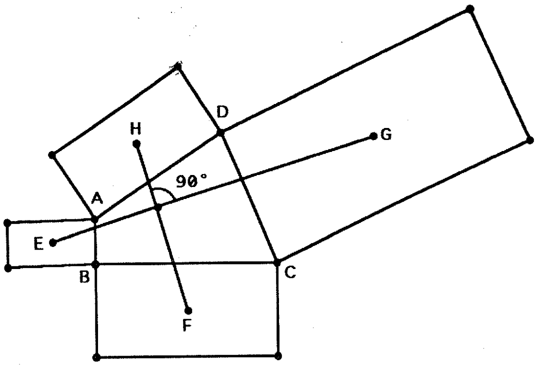


Figure 109

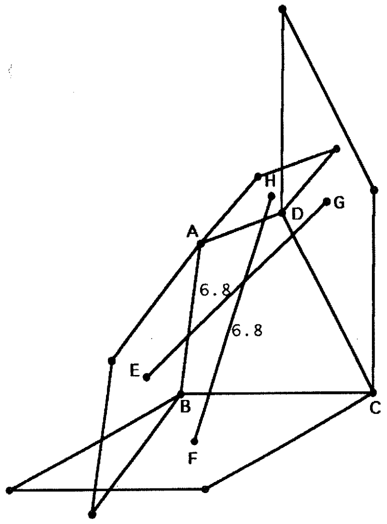


Figure 110

All four generalizations can easily be investigated dynamically with software like *Sketchpad* or confirmed in general by the property checker of *Cabri-Geometre.* It is furthermore important to stress that these generalizations were first experimentally explored and confirmed before the author started looking for proofs (explanations). So this is clearly a case where quasi-empirical conviction preceded and motivated formal proof (compare Chapter 3).

Note that just as squares are the intersection of the rectangles and rhombi, so Von Aubel's theorem is the intersection of these two dual generalizations. As far as the author could ascertain, these two generalizations, as well as their respective specializations with regard to parallelograms, have not been published before.

Proofs

First result

Consider Figure 111. Since a half-turn clearly maps the figure as a whole onto itself, it follows that EFGH is a parallelogram. We shall now prove the result by showing ΔEBF congruent to ΔEAH. If we let $\angle HAD = y$, then $\angle XAH = 90^\circ - y = \angle EBA = \angle EAB$ and $\angle FBC = y = \angle EAY$. If we let $\angle ABC = x$, then $\angle BAD = 180^\circ - x$. Therefore:

$$\angle YAX = 360^\circ - (\angle YAB + \angle BAD + \angle XAD) = 360^\circ - \left(90^\circ + 90^\circ + 180^\circ - x\right) = x.$$

We now have $\angle EBF = \angle EBA + \angle ABC + \angle FBC = 90^\circ - y + x + y = 90^\circ + x$, as well as $\angle EAH = \angle EAY + \angle YAX + \angle XAH = y + x + 90^\circ - y = 90^\circ + x$.

Therefore $\angle EBF = \angle EAH$, from which follows that triangles EBF and EAH are congruent $(s, \angle, s)$ and that EH = EF. But a parallelogram with two adjacent sides equal, is a rhombus.

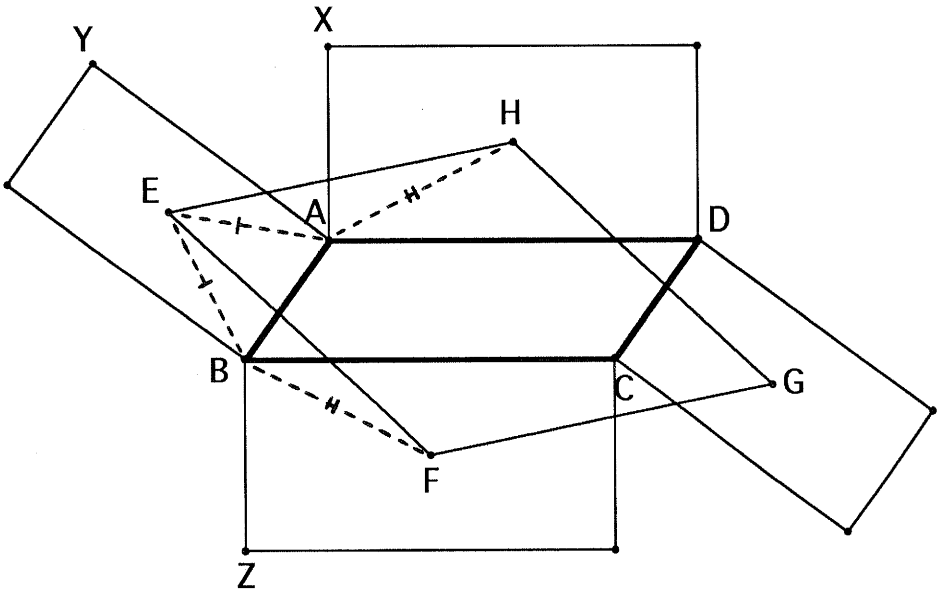


Figure 111

Second result

Consider Figure 112. Since a half-turn clearly maps the figure as a whole onto itself, it follows that EFGH is a parallelogram. We shall now first prove that triangles EAH and EBF are similar.

If we let $\angle HAD = y$, then $\angle EAB = y$ and $\angle EBA = 90^\circ - y = \angle FBC$. If we let $\angle ABC = x$, then $\angle BAD = 180^\circ - x$. We now have:

$$\angle EAH = \angle EAB + \angle BAD + \angle HAD = y + 180^\circ - x + y = 180^\circ - x + 2y \text{ and}$$
$$\angle EBF = 360^\circ - (\angle EBA + \angle ABC + \angle FBC) = 360^\circ - \left(90^\circ - y + 90^\circ - y + x\right) = 180^\circ - x + 2y.$$

Therefore $\angle EAH = \angle EBF$, but from the similarity of the rhombi EA/EB = AH/BF. Thus, triangles EAH and EBF are similar so that $\angle AEH = \angle BEF$. But $\angle AEB = 90^\circ = (\angle AEH) + \angle HEB = (\angle BEF) + \angle HEB = \angle FEH$. But a parallelogram with a right angle is a rectangle.

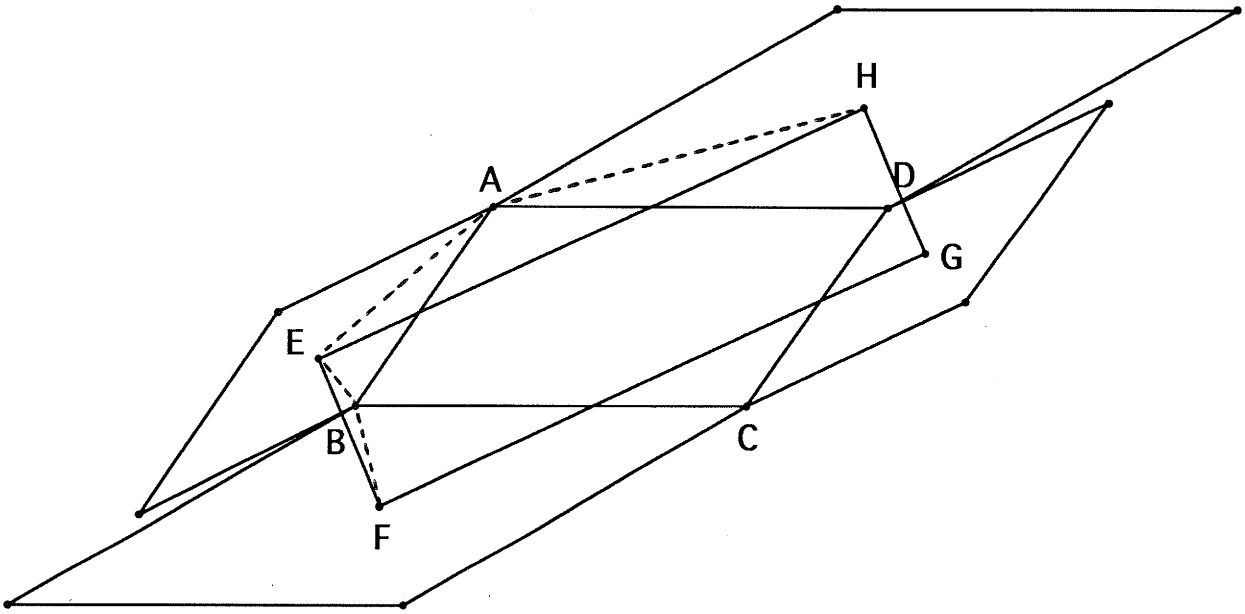


Figure 112

The following two proofs for the third and fourth results are generalizations of the proof by Yaglom (1962:94-97) for Von Aubel's theorem. Contrary to normal convention, we shall here first present proofs for the third and fourth results and then follow them by necessary lemmas utilized in the proofs. (Actually, this is more in accordance with the way in which proofs are usually discovered).

Third result

Consider Figure 113. Let $\angle AM_1B = 2x$, then the sum of the the four rotations about the points M_1, M_2, M_3 and M_4, respectively through angles of $2x$, $180^\circ - 2x$, $2x$ and $180^\circ - 2x$, clearly carries the vertex A of the quadrilateral into itself. It follows that this sum of four rotations is the identity transformation. But the sum of the rotations about M_1 and M_2 is equivalent to a half-turn about O_1 - the vertex of a right triangle $O_1M_1M_2$ with $\angle M_1O_1M_2 = 90^\circ$, since $\angle O_1M_1M_2 = x$ and $\angle O_1M_2M_1 = 90^\circ - x$ (see Lemma 1).

Similarly the sum of rotations about M_3 and M_4 is a half-turn about the vertex O_2 of a right triangle $O_2M_3M_4$ with $\angle M_3O_2M_4 = 90^\circ$. From the fact that the sum of the half-turns about O_1 and O_2 is the identity transformation it clearly follows that these two points coincide.

Since triangles $O_1M_1M_2$ and $O_2M_3M_4$ are similar, we can obtain triangle $O_1M_1M_3$ from triangle $O_1M_2M_4$ by the spiral similarity $(k, 90^\circ)$ about the point $O_1 = O_2$ (a rotation through 90° about the point $O_1 = O_2$ followed by a dilation (enlargement or reduction) from the same point with factor $k = O_1M_1 / O_1M_2 = O_2M_3 / O_2M_4$). Therefore the corresponding segments M_1M_3 and M_2M_4 of triangles $O_1M_1M_3$ and $O_1M_2M_4$ are perpendicular.

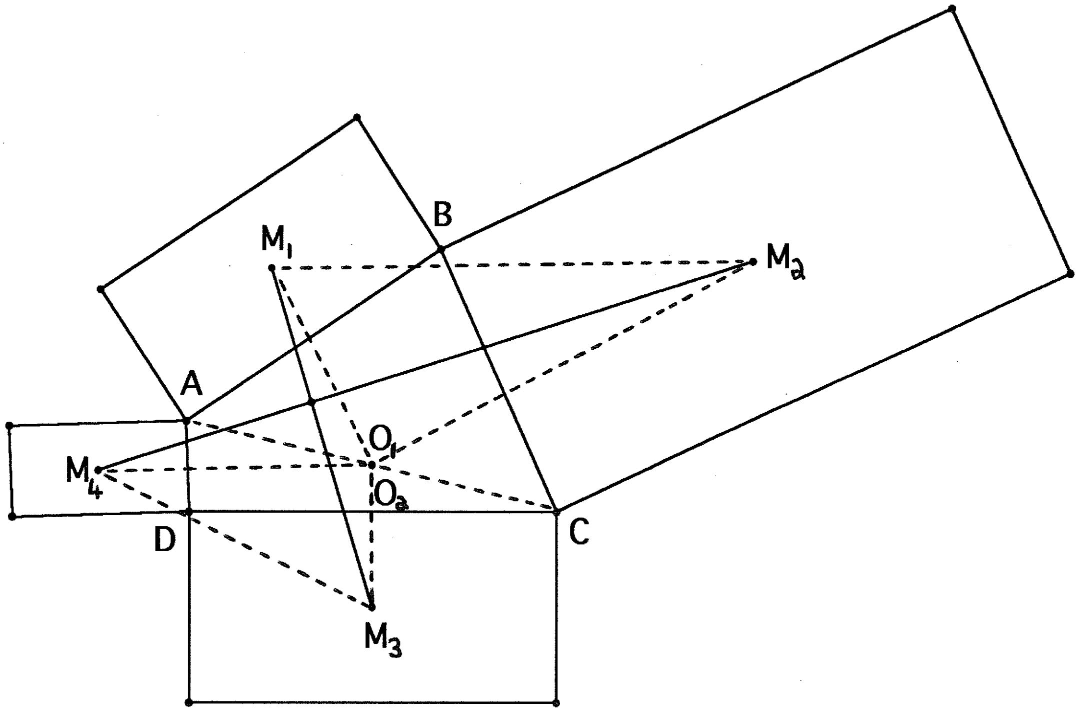


Figure 113

Fourth result

Consider Figure 114. The sum of the four spiral similarities $(k,90^\circ)$, $(1/k,90^\circ)$, $(k,90^\circ)$ and $(1/k,90^\circ)$ about the points M_1, M_2, M_3 and M_4 clearly carries the vertex A of the quadrilateral onto itself. (From the similarity of the rhombi we have $k = M_1B/M_1A = M_2B/M_2C = M_3D/M_3C = M_4D/M_4A$). It follows that this sum of four spiral similarities is the identity transformation. But the sum of the spiral similarities about M_1 and M_2 is equivalent to a half-turn about O_1 - the vertex of an isosceles triangle $O_1M_1M_2$ with $\angle O_1M_1M_2 = \angle O_1M_2M_1 = \arctan(1/k)$ and $\angle M_1O_1M_2 = 180^\circ - 2\arctan(1/k)$ (See Lemma 2).

In the same way the sum of the two spiral similarities about M_3 and M_4 is a half-turn about the vertex O_2 of an isosceles triangle $O_2M_3M_4$. From the fact that the sum of the half-turns about O_1 and O_2 is the identity transformation it follows that these two points coincide.

Since triangles $O_1M_1M_2$ and $O_2M_3M_4$ are similar, we can obtain triangle $O_1M_1M_3$ from triangle $O_1M_2M_4$ by a rotation through $180^\circ - 2\arctan(1/k)$ about the point $O_1 = O_2$. Therefore the corresponding segments M_1M_3 and M_2M_4 of triangles $O_1M_1M_3$ and $O_1M_2M_4$ are equal.

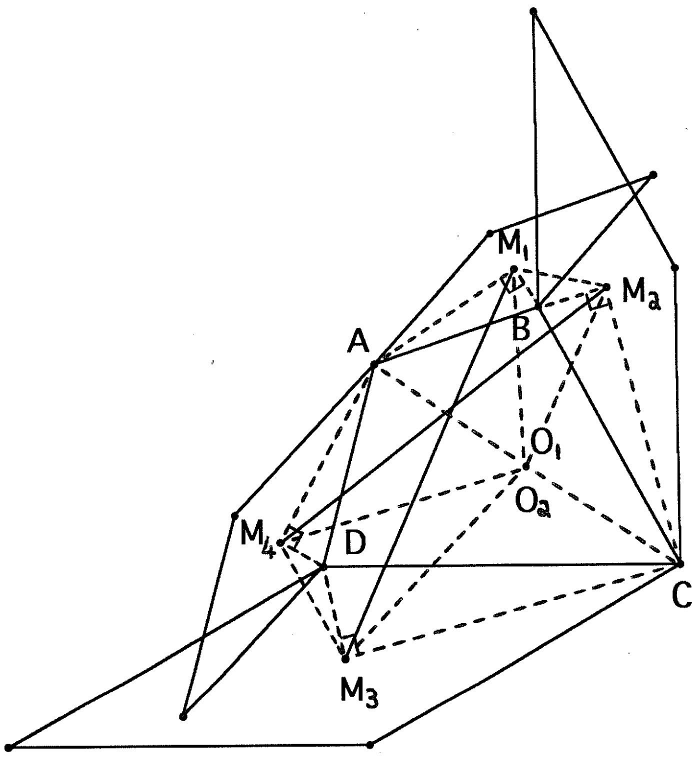


Figure 114

Lemma 1

"The sum of two rotations with centres O_1 and O_2 through angles α and β respectively is a rotation through the angle $\alpha+\beta$ around some centre O."

Consider Figure 115. The sum of the two rotations carries the centre O_1 of the first into a point O_1' such that $O_1'O_2 = O_1O_2$ and $\angle O_1O_2O_1' = \beta$. (The first rotation leaves O_1 in place, and the second carries O_1 into O_1'). The sum of the two rotations also carries a point O_2' into O_2 such that $O_2'O_1 = O_2O_1$ and $\angle O_2'O_1O_2 = \alpha$. (The first rotation carries O_2' into O_2 and the second leaves O_2 in place).

From this it follows that the centre O we are seeking is equidistant from O_2 and O_2' and from O_1' and O_1; consequently it can be found as the point of intersection of the perpendicular bisectors l_1 and l_2 of the segments O_2O_2' and $O_1'O_1$ respectively. But from Figure 115, it is clear that l_1 passes through O_1 and $\angle OO_1O_2 = \alpha/2$, and that l_2 passes through O_2 and $\angle O_1O_2O = \beta/2$. The lines l_1 and l_2 are completely determined by these conditions; therefore we have found the desired centre of rotation O. Finally, it should now be clear that $\angle O_2OO_2' = \alpha+\beta = \angle O_1OO_1'$ so that a rotation of $\alpha+\beta$ around O maps O_2' onto O_2 and O_1 onto O_1'.

Lemma 2

"The sum of two spiral similarities $(k,90^\circ)$ and $(1/k,90^\circ)$ around centres O_1 and O_2 is a half-turn around a centre O, the vertex of an isosceles triangle O_1O_2O with $OO_2 = OO_1$."

Consider two spiral similarities $(k,90^\circ)$ and $(1/k,90^\circ)$ from the respective centres O_1 and O_2 (see Figure 116). The first spiral similarity leaves O_1 in place, and the second carries O_1 into O_1' by the rotation of 90° and then into O_1'' by the dilation with factor $1/k$. The sum of the two spiral similarities therefore carries the centre O_1 into a point O_1'' such that $O_1''O_2 = (O_1O_2)/k$.

The first spiral similarity carries a point O_2' into O_2'' by the rotation of 90° and then into O_2 by the dilation with factor k. The second spiral similarity leaves O_2 in place. (Therefore $O_2''O_1 = O_1O_2' = (O_1O_2)/k$). The sum of the two spiral similarities therefore carries a point O_2' into O_2.

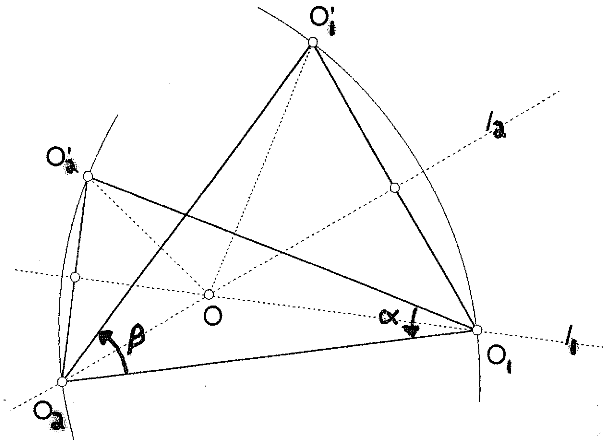


Figure 115

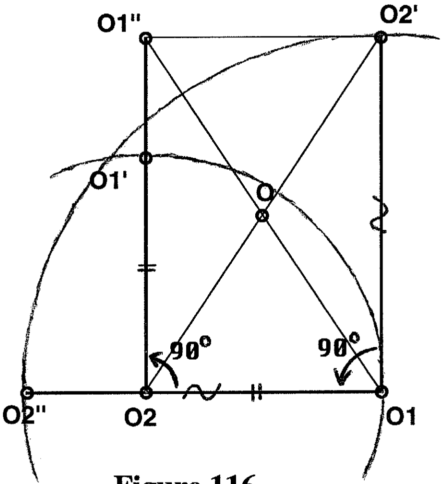


Figure 116

Thus $O_2O_1O_2'O_1''$ is a rectangle and the required point of half-turn rotation equivalent to the two spiral similarities is O, the intersection of the diagonals of this rectangle. Furthermore, triangle O_1O_2O is an isosceles triangle and $\tan(\angle O_2O_1O_1'') = 1/k$ and therefore $\angle O_2O_1O = \angle OO_2O_1 = \arctan(1/k)$ and $\angle O_2OO_1 = 180^\circ - 2\arctan(1/k)$.

Looking back

Analysis of the above proof of Lemma 2 shows that it is immediately generalizable as follows: "The sum of two spiral similarities (k,α) and $(1/k,\beta)$ around centres O_1 and O_2 is a half-turn around a centre O, the intersection of the diagonals of a parallelogram $O_2O_1O_2'O_1''$ if $\alpha + \beta = 180°$".

Questions and Problems 8

1. Can you still further generalize the two dual Von Aubel generalizations by appropriately constructing for example similar isosceles trapezia and kites on the sides, or other more general quadrilaterals?

2. Can you prove the two dual Von Aubel generalizations using vectors?

Some further "what-if" questions

The reader may now wish to explore further questions like the following:

- What happens if ABCD is concave or crossed? Are the results still true? If so, are the above proofs still valid?

- What happens if the similar rectangles and rhombi are constructed "*inward*"? Are the results still true? If so, are the above proofs still valid?

- What happens if the results are specialized by letting some of the vertices of ABCD coincide to produce a triangle or line segment?

- What happens if instead of a base quadrilateral, a hexagon or octagon is chosen? Can the results be generalized to $2n$-gons?

- Can the results be generalized to three dimensions, or in general, to n dimensions?

Epilogue

In conclusion of this brief journey through some aspects of geometry, it is hoped that the reader has had some opportunity to share and experience, together with the author, in the adventure and excitement of discovery and proof. Hopefully the reader will also be stimulated into continuing *adventuring* through mathematics on his/her own.

As mentioned in several places before, the particularly appealing thing about mathematics is that solutions to questions almost invariably lead to new questions, thus providing an endless source for exploration and adventure. The only possible limitation is the availibility and finiteness of time, as Donald Knuth's characters so poignantly say at the end of his book **Surreal Numbers** (1974:111):

"Alice (falling into his arms): *Bill! Every discovery leads to more, and more!*
Bill (glancing at the sunset): *There are infinitely many things to do ... and only a finite amount of time ...*"

Solutions 1

1. (a) The rhombi. A kite with opposite sides parallel must have opposite sides equal (parallelogram property). But a kite has adjacent sides equal, and therefore all sides are equal.

(b) The rectangles. A quadrilateral with equal adjacent angles (isosceles trapezium property), as well as equal opposite angles (parallelogram property) must have all angles equal ($=90°$).

(c) The rhombi. Prove that a kite with one pair of opposite sides parallel is a rhombus. Consider Figure 1.1. Angles BAC and DAC are equal (kite property), but if (say) AD//BC, we have $\angle BCA = \angle DAC$ which implies $\angle BAC = \angle BCA$. Therefore triangle ABC is isosceles, and from symmetry around AC we have all sides equal.

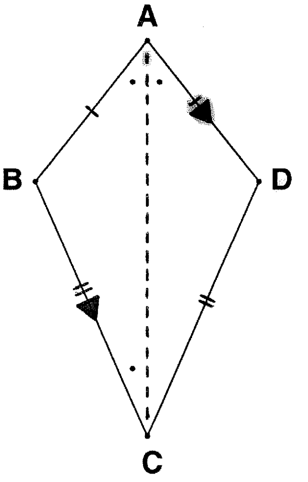


Figure 1.1

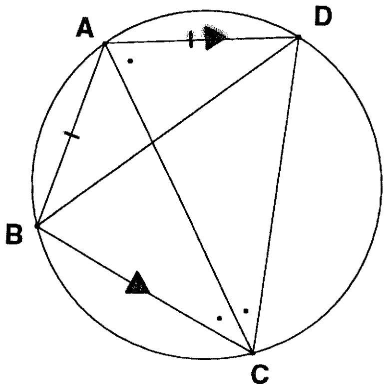


Figure 1.2

(d) The trilateral trapezia. Prove that a skew cyclic quadrilateral with one pair of opposite sides parallel is a trilateral trapezium (see Figure 1.2). If AD//BC, then $\angle BCA = \angle CAD$. But since chords AB and AD are equal, we have $\angle BCA = \angle ACD$. Therefore $\angle CAD = \angle ACD$ which implies AD=CD and that we have an isosceles trapezium with three equal sides as indicated.

(e) The squares. If a right kite has three sides equal (trilateral trapezium property) it must be a rhombus (all sides equal). But a rhombus with a right angle is a square.

(f) The trilateral trapezia. By definition, we have the skew trapezia as the intersection between the trapezia and skew kites and the skew cyclic quads as the intersection between the cyclic quads and skew kites. But the intersection between the skew trapezia and the skew cyclic quads is the trilateral trapezia as shown in 1(d).

(g) The squares. The intersection between the trapezia and cyclic quads is the isosceles trapezia (see text). But an isosceles trapezium with one pair of opposite angles equal (kite property) must be a rectangle (all angles equal). But a rectangle with one pair of adjacent sides equal (kite property) is a square.

2. See Figure 1.3.

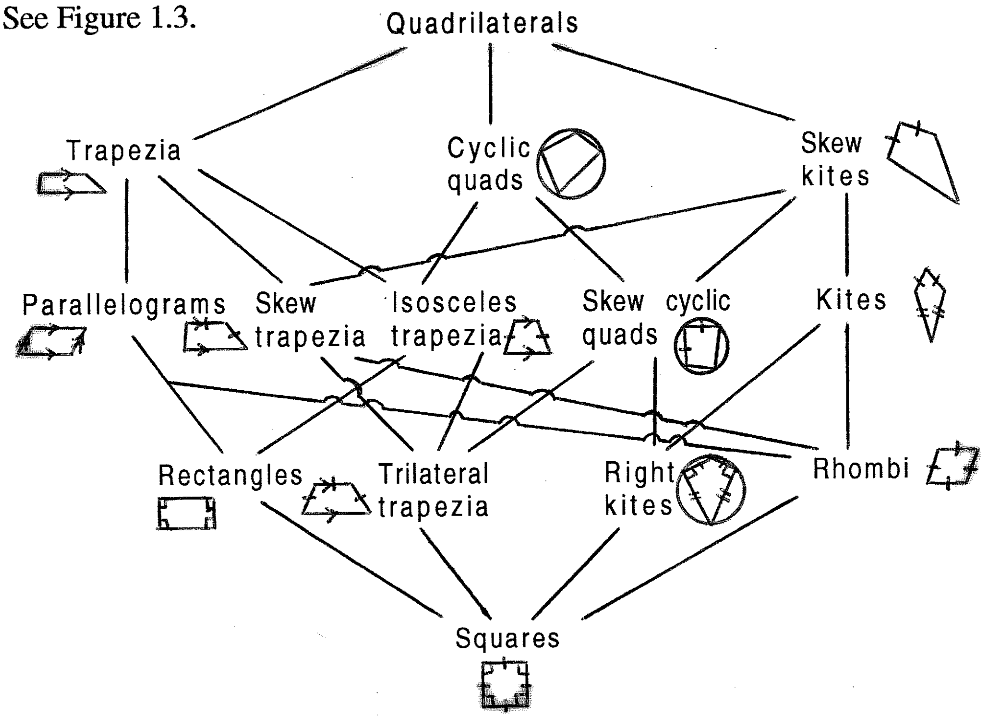


Figure 1.3

3. The diagonals of a trilateral trapezium bisect the two (base) angles adjacent to the fourth (unequal) side into equal halves. Consider Figure 1.4. Angles 3 and 8 are equal since ΔABD is isosceles. But $\angle 8 = \angle 5$ since AD//BC and therefore $\angle 3 = \angle 5$; that is, diagonal DB bisects $\angle B$. In the same way we can show that diagonal AC bisects $\angle C$ and since $\angle B = \angle C$ they are bisected in equal halves. (Note that it also follows from this argument that one of the (base) angles of a skew trapezium will always be bisected by a diagonal).

4. The diagonals of an isosceles trapezium are equal. Consider Figure 1.5. AC=BD since ABCD is cyclic and $\angle B = \angle C$ (equal angles on the circumference are subtended by equal chords). Or alternatively, triangles ABC and DCB are congruent $(s, \angle, s)$.

5. Given a quadrilateral with the properties as shown in Figure 1.6. Construct DO parallel to AB with O on BC. Then ABOD is a parallelogram which implies AB=DO. But AB=DC and therefore DO=DC which implies $\angle DOC = \angle C$. But $\angle DOC = \angle B$ since AB//DO and therefore $\angle B = \angle C$, and since $\angle A$ and $\angle D$ are the supplements of $\angle B$ and $\angle C$ it follows that $\angle A = \angle D$.

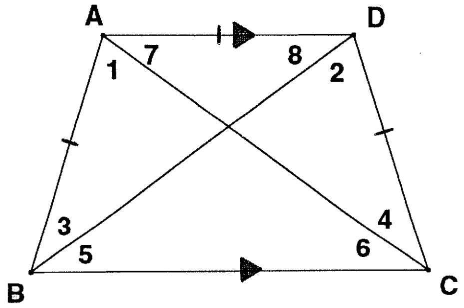


Figure 1.4

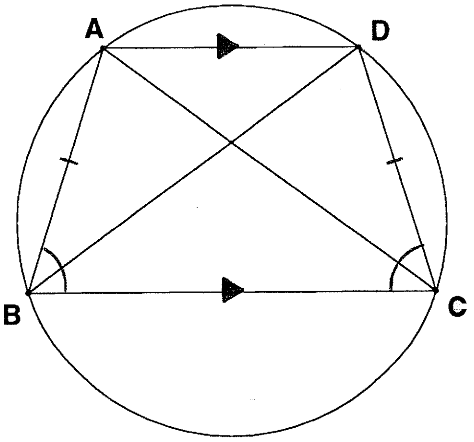


Figure 1.5

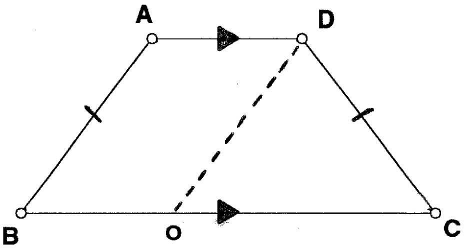


Figure 1.6

6. One's first natural reaction may be one of "*monster-barring*", i.e. to reject this figure as a quadrilateral because it does not conform to our usual representation of a quadrilateral. Furthermore, one might argue that it is not a quadrilateral because the sum of its angles are not 360°. One might therefore try to define a quadrilateral in such a way that figures like these are excluded.

   However, one may also choose to consider this figure as a quadrilateral, which is what we will do in the next chapter.

7. It becomes more and more difficult to draw Venn-diagrams as the number of figures and their intersections increase. That is why the classification scheme in the text was chosen since it more easily handles complex classifications.

Solutions 2

1. No, it's impossible to prove **all** the properties of a concept, since that would inevitably lead to a **circular** argument. For example, let's assume for simplicity's sake that a parallelogram has just the following two properties:
(a) opposite sides parallel
(b) opposites sides equal.

If we now try to prove both properties, we could for instance start out by proving (a) by the assumption of (b). However, in order to prove (b) we would have to assume (a) which gives us a circular argument (b) ⇒ (a) ⇒ (b). Similarly, we get a circular argument if we set out by proving (b) first. In order to avoid such circular reasoning, one of the properties must therefore be *accepted without proof* as the **definition** for the concept involved.

[The reader may wish to verify that circular arguments inevitably arise even if all the properties of a parallelogram are considered, although the circular argument may be more difficult to recognize as it may involve more than two properties, eg. (a)⇒ (b)⇒ (c)⇒ (a)].

2. (a) Necessary and sufficient. (See *Questions & Problems 1*, no.1(c)).
(b) Sufficient but not necessary. Three equal sides are not necessary; one pair of opposite sides equal is necessary and sufficient (See Chapter 1).
(c) Necessary and sufficient (See Chapter 1).
(d) Necessary but not sufficient. An isosceles trapezium for example also has equal diagonals.
(e) Necessary but not sufficient. It is possible to construct a quadrilateral with equal diagonals as shown in Figure 2.1 that is not an isosceles trapezium.

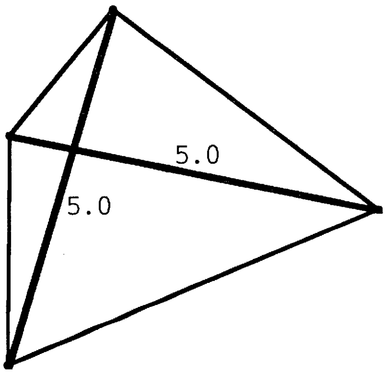


Figure 2.1

(f) Necessary but not sufficient. An angle quad (see Figure 20) also has equal opposite angles.

(g) Necessary but not sufficient. In fact, all corresponding angles equal is not even sufficient for the similarity of two quadrilaterals. For example, in Figure 2.2 two rectangles are shown which obviously have corresponding angles equal, but are not similar. (Under what (minimum) conditions are two quadrilaterals similar? Investigate.)

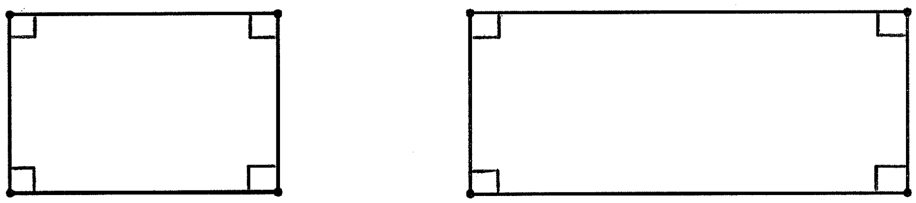


Figure 2.2

(h) Necessary and sufficient. Consider Figure 2.3. Connect A with C, then $\angle DAC = \angle BCA$. Therefore triangles ABC and CDA are congruent $(\angle, \angle, s)$ so that AD=CB, AB=CD, etc.

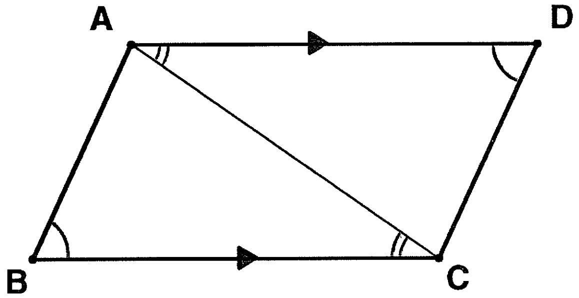


Figure 2.3

(i) Necessary but not sufficient. An isosceles trapezium also complies to this condition.

(j) This is a tricky one. Consider Figure 2.4. Draw BX ⊥ AD, DY ⊥ BC and join B and D. Then it's easy to first prove triangles AXB and CYD congruent $(\angle, \angle, s)$, and then triangles BXD and DYB congruent $(90°, s, s)$. Therefore AX=CY and XD=YB from which we have AD=AX+XD=CY+YB=CB. In other words, ABCD is a parallelogram.

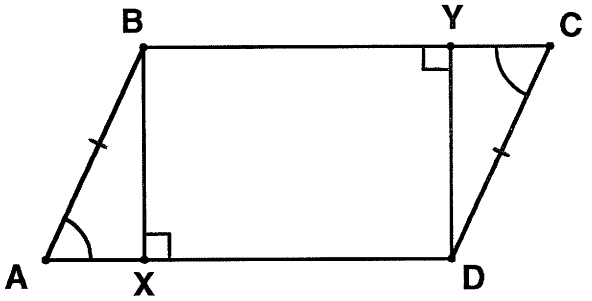


Figure 2.4

Do you find this argument satisfactory? Is this condition therefore necessary and sufficient? (See *Solutions 2(continued)*).

(k) Necessary and sufficient. Consider Figure 2.5. Suppose O is the point of half-turn symmetry. Rotating ABCD around this point through 180° must therefore map AB onto CD and BC onto DA. Therefore opposite sides are equal. Furthermore, since the rotation is through 180°, AOC and BOD must be straight lines with AO=OC and BO=OD. Since $\angle BAC$ and $\angle ADB$ respectively map onto $\angle DCA$ and $\angle CBD$, we also have opposite sides parallel. This is another good example of a *deductive-economical* definition which enables the almost immediate derivation of the other properties from it.

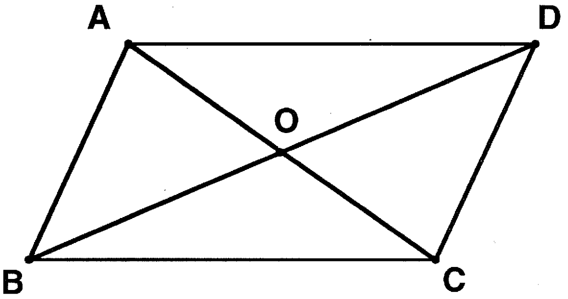


Figure 2.5

Note: (1) A figure has *half-turn symmetry*, if and only if, rotated through 180°, it maps onto itself. (2) In the plane, half-turn symmetry is also equivalent to the different concept of *point-symmetry*. A figure is point symmetric around a point O, if and only if, for every point P on the figure there exists a corresponding point P' also on the figure under a reflection in O (see Figure 2.6).

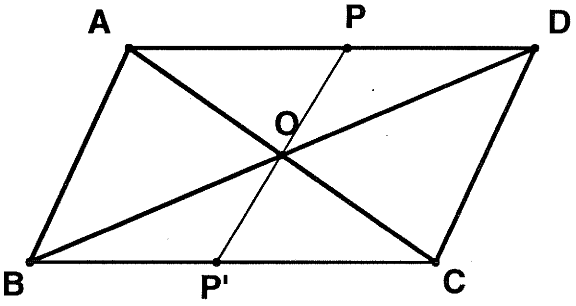


Figure 2.6

(l) Necessary and sufficient. Consider Figure 2.7. Reflecting ABCD in x, angles A and B respectively map onto D and C which implies they are equal. Reflecting ABCD in y, angles A and D respectively map onto B and C, and therefore all angles are equal.

(m) Necessary and sufficient. Consider Figure 2.8. Reflecting ABCD in x, we have AB=DC, $\angle A = \angle D$ and $\angle B = \angle C$. Reflecting ABCD in y, we have AB=AD, DC=BC and $\angle B = \angle D$. Therefore all sides and angles are equal.

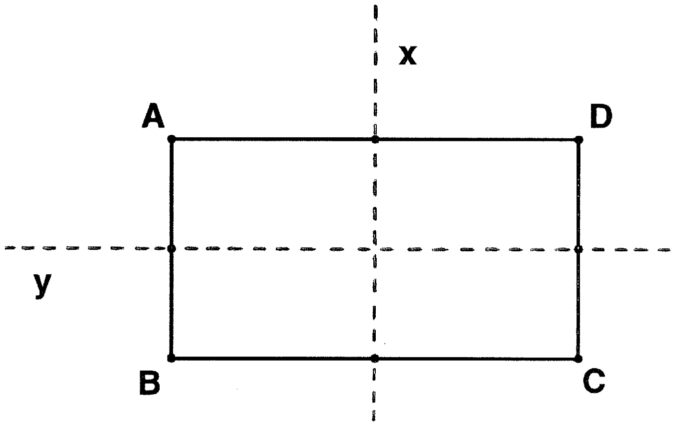


Figure 2.7

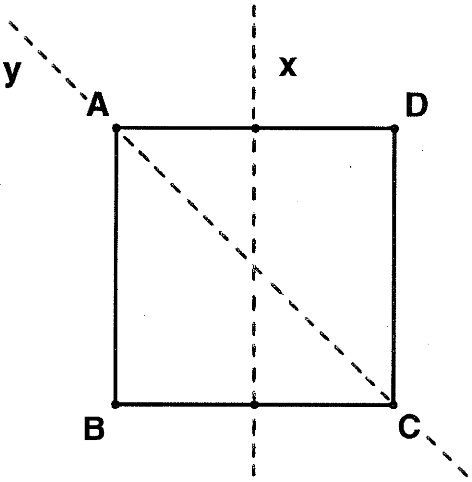


Figure 2.8

(n) Sufficient but not necessary. Only one pair of opposite angles supplementary is necessary for a quadrilateral to be cyclic, not two. (See text in Chapter 1 for a proof that a cyclic quad with one pair of opposite sides parallel is an isosceles quad).

(o) Necessary but not sufficient. Consider Figure 2.9.

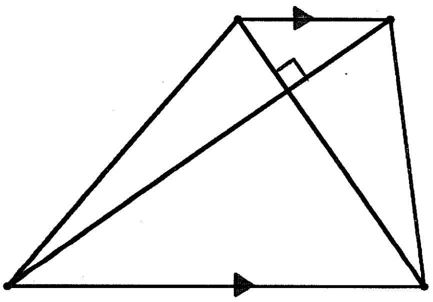


Figure 2.9

3. (a) This is a circular definition - we cannot define something in terms of itself.

(b) It is impossible for any quadrilateral to have all its side parallel - at most, two opposite sides can be parallel.

(c) It includes only the squares.

(d) Isosceles trapezia do not necessarily have perpendicular diagonals nor are all quadrilaterals with perpendicular diagonals, isosceles trapezia (e.g. see Figure 20).

4. (i) An isosceles trapezium is a quadrilateral with one pair of opposite sides parallel.

(ii) An isosceles trapezium is a cyclic quadrilateral with equal diagonals, one pair of opposite sides parallel and two pairs adjacent sides equal.

(iii) An isosceles trapezium is a quadrilateral with and axis of symmetry through one pair of opposite sides. An isosceles trapezium is a quadrilateral with two pairs of adjacent angles equal. An isosceles trapezium is a quadrilateral with one pair of opposite sides parallel and the other pair equal (but not parallel).

6. (i) A rectangle is a quadrilateral with all angles equal, but not all sides.

(ii) A rhombus is a quadrilateral with all sides equal, but not all angles.

(iii) A cyclic quadrilateral is a quadrilateral with opposite angles supplementary, but no axes of symmetry through opposite sides (excludes isosceles trapezia) or angles (excludes kites).

(iv) A bisecting quadrilateral is a quadrilateral with one of the diagonals bisecting the other, but no axes of symmetry through opposite angles (excludes kites) or a point of symmetry (excludes parallelograms).

7. One way to extend the notion of *"internal"* angles to crossed quadrilaterals is by first analysing and defining the notion of internal angles for convex and concave quadrilaterals and then consistently applying that definition to crossed quadrilaterals. (This is a strategy often used in mathematics when extending certain concepts beyond their original domain).

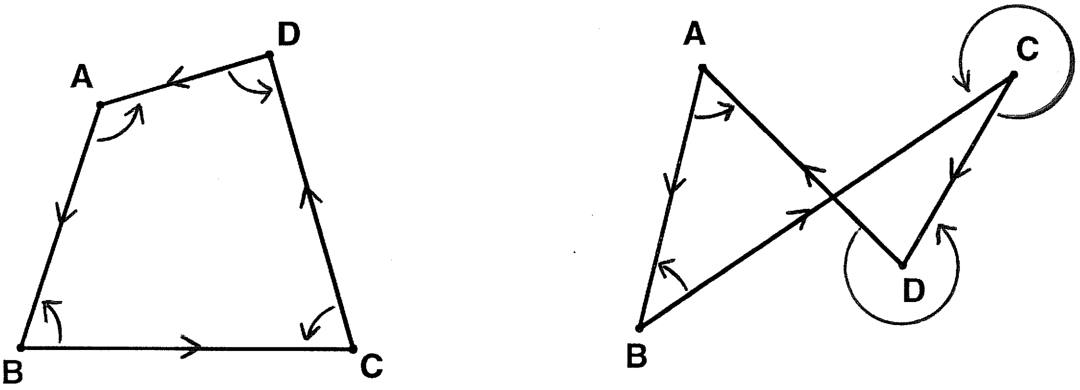


Figure 2.10

Suppose we walk from A to B, B to C, etc. around the perimeter of the convex quadrilateral shown in Figure 2.10. The internal angle at each vertex will then be the angle through which the *next side* must be rotated *anti-clockwise* (with the vertex as rotation centre) to coincide with the *previous side*. (Note if the figure is alphabetically labelled in a clockwise direction the angle through which the next side must be rotated to coincide with the previous side should also be clockwise). In the same way we can now determine the internal angles of the crossed quadrilateral by walking around the perimeter as shown. This therefore leads us to the surprising conclusion that the two **reflexive** angles at C $(360° - \angle BCD)$ and D $(360° - \angle ADC)$ are the "*internal*" angles of the crossed quadrilateral ABCD.

Now find the sum of the interior angles of a crossed quadrilateral. (See solution in *Solutions 2 (continued)*).

(b) Again working from the convex and concave cases, we can define the following for a crossed quadrilateral ABCD (see Figure 2.10):
opposite angles: $\angle A$ and $\angle C$; $\angle B$ and $\angle D$
opposite sides: AB and CD; AD and BC.

8. A convex quadrilateral has no diagonals outside (or no reflexive angles).
A concave quadrilateral has one diagonal outside (or one reflexive angle).
A crossed quadrilateral has both diagonals outside (or two reflexive angles).

9. (a) Bisecting quad - only convex and concave (See Figure 2.11). (Crossed ones are not possible since both diagonals fall outside the figure and their extensions would therefore also meet outside the diagonals, and it would therefore not be possible for the one to bisect the other).

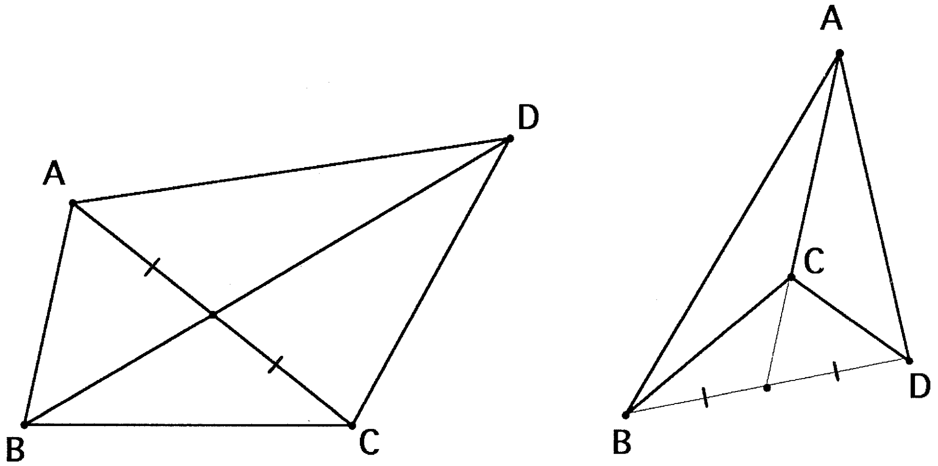


Figure 2.11

(b) Perpendicular quad - convex, concave and crossed (See Figure 2.12).

(c) Trapezium - convex and crossed (see Figure 2.13). (Concave trapezia are not possible since co-interior angles must be supplementary in a simple closed trapezium and it is therefore impossible for an angle to be reflexive).

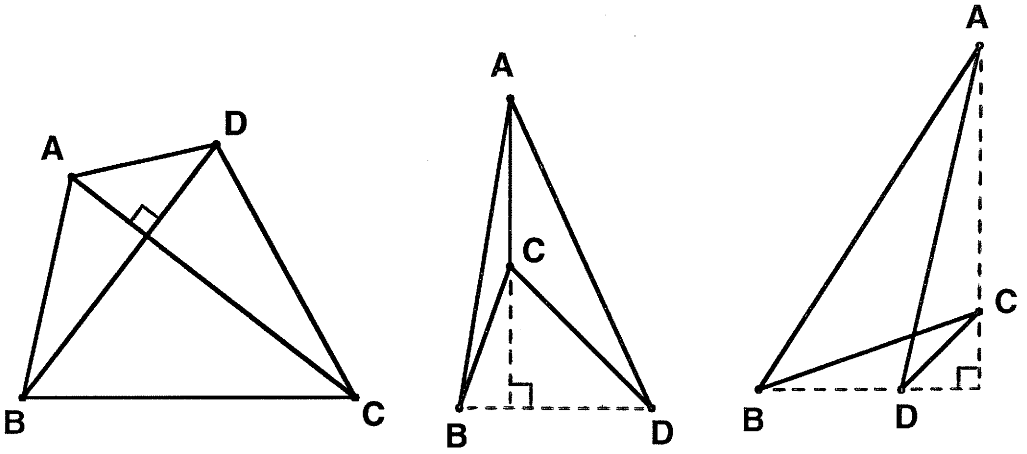


Figure 2.12

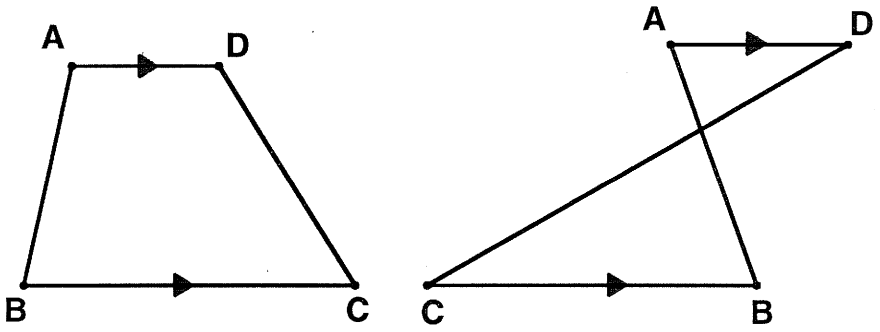


Figure 2.13

(d) Cyclic quad - convex and crossed (see Figure 2.14). (Concave ones are not possible since opposite angles must be supplementary in a simple closed cyclic quad, and it is therefore impossible for an angle to be reflexive).

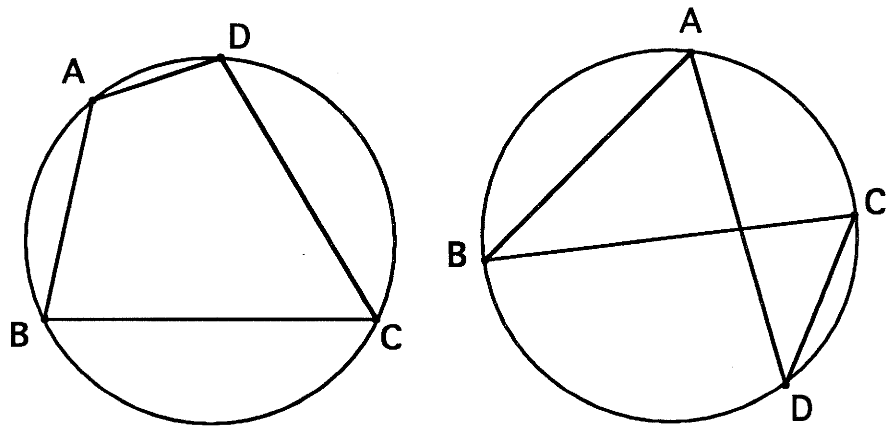


Figure 2.14

(e) For the parallelograms it depends on the definition we choose. If we choose the traditional one of opposite sides parallel, then only convex ones are possible since a crossed quadrilateral cannot have both pairs of opposite sides parallel (one pair crosses each other).

However, if we define a parallelogram as a quadrilateral with one pair of opposite sides parallel and equal (or having half-turn symmetry), then we can obtain both convex and crossed parallelograms (see Figure 2.15). In the latter case we have the diagonals AC and BD equal and parallel to each other, opposite sides AB and CD not equal and bisecting each other, and the two pairs of opposite angles not equal.

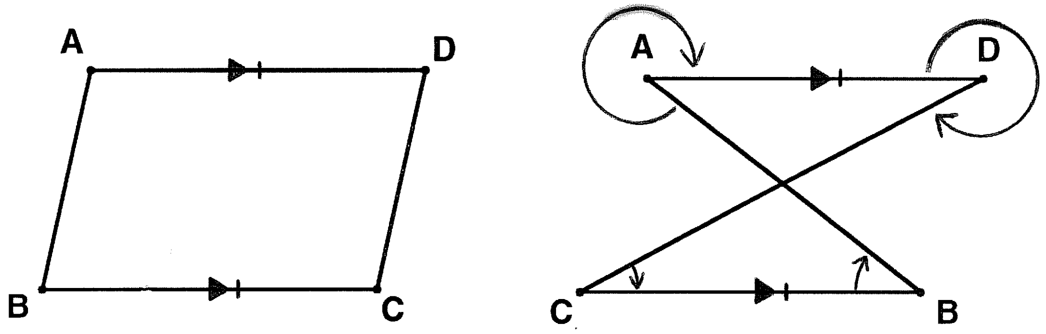


Figure 2.15

Suppose we agreed to accept the latter definition of a parallelogram. This would imply the partitioning of the parallelograms into two distinct classes with different properties and would affect the formulation of several of our traditional theorems. For example, we could then no longer say that *"the diagonals of **any** parallelogram bisect each other"* but would have to restrict it to only convex parallelograms.

However, let us agree to the first definition of parallelograms which is in line with the usual convention.

(f) Right kite - convex (see Figure 5 in text). (Concave ones are not possible since it is impossible to have concave cyclic quads. Similarly we cannot have crossed ones since it is impossible to have a crossed kite).

(g) Rhombus - convex. (Concave ones are not possible since a rhombus is a parallelogram and it is impossible to have a concave parallelogram).

(h) Rectangle - convex. (Crossed rectangles are not possible since a crossed quadrilateral cannot have all its angles equal).

10. (b) Perhaps surprisingly, for crossed cyclic quads three equal sides are necessary and sufficient. The difference is that for a crossed cyclic quad (in contrast to a convex cyclic quad) one pair of equal opposite sides is not sufficient for it to be an isosceles

trapezium (see Figure 2.16a where none of its pairs of opposite sides (AB & CD; BC & AD) are parallel, and diagonals BD and AC are not equal). However, if the pair of equal opposite sides intersect as shown in Figure 2.16b, then it can be shown that the other pair of opposite sides are parallel. But as shown in Figure 2.16c, a crossed cyclic quad with three equal sides has one pair of equal opposite sides intersecting. (Two different cases are given, namely, ABCD with AB=AD=BC and ABCD' with AB=BC=CD').

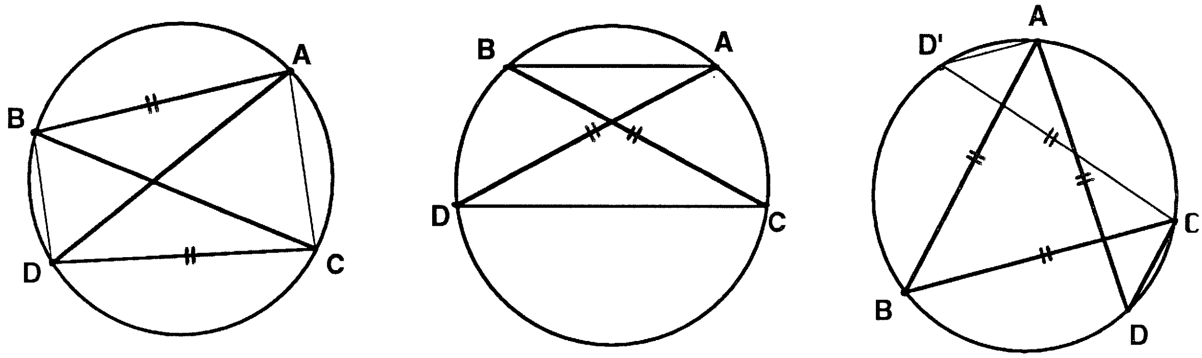


Figure 2.16

(k) Necessary but not sufficient. If we confine the notion of parallelogram to only convex ones, a point of half-turn symmetry is not sufficient since it includes a crossed case (See Solution 9(e) above). The condition of half-turn symmetry is of course a necessary and sufficient condition for the *convex* quadrilaterals, but not for quadrilaterals in general.

(n) Note that the condition of one pair of opposite angles supplementary and one pair of opposite sides parallel is necessary and sufficient for a quadrilateral to be a *convex* isosceles trapezium, since a crossed quadrilateral cannot have opposite angles supplementary.

11. (a) Yes. It is only necessary to consider convex and concave cases since a bisecting quad cannot be a crossed quad. Using Figure 2.17 check that the proof given in the text (Figure 16) also applies in the concave case.

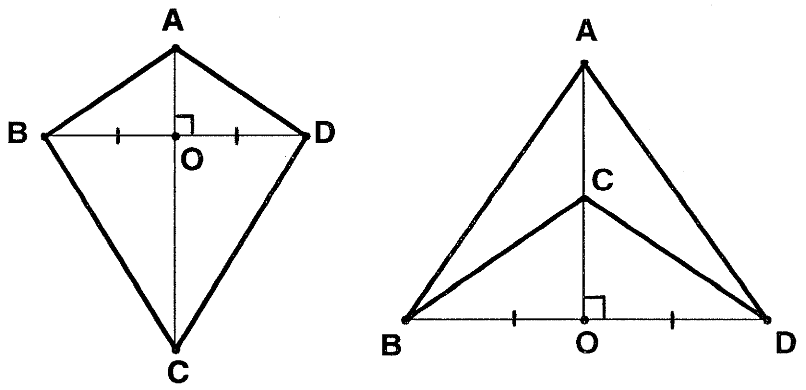


Figure 2.17

(b) No, not necessarily as shown by the counter-example in Figure 2.18a. However, if it is given that the equal sides are *adjacent* to the non-bisected diagonal as shown in Figure 2.18b-c, we do obtain a kite (the proof is left to the reader).

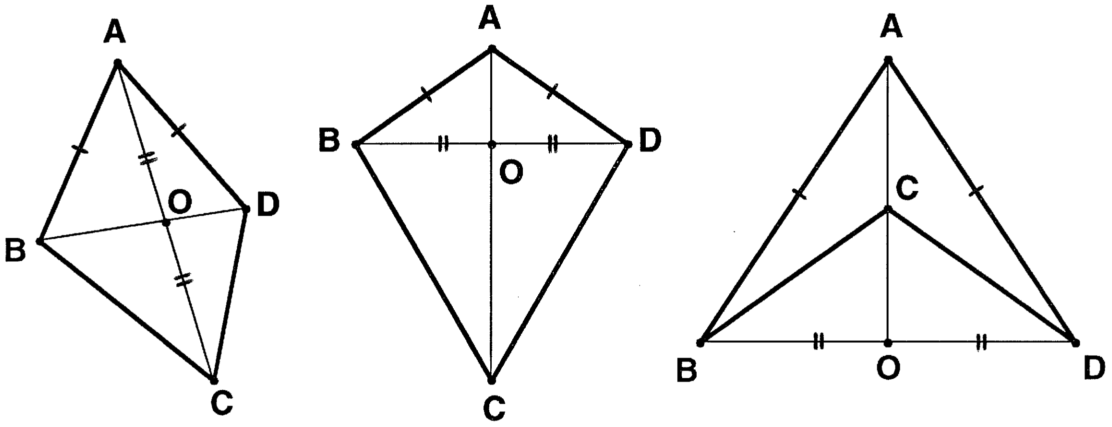


Figure 2.18

(c) No, not necessarily as shown by the counter-example in Figure 2.19a. However, if it is given that none of the equal opposite angles are included by the pair of equal adjacent sides, then it is true. It is only necessary to consider convex and concave cases since an angle quad cannot be crossed (one pair of opposite angles cannot be equal in the latter). Consider Figure 2.19b-c.

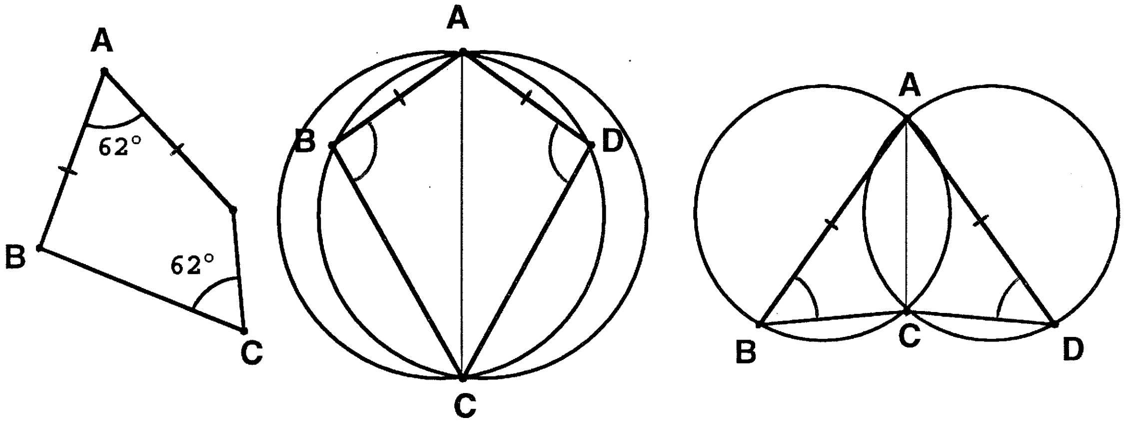


Figure 2.19

Consider firstly any circle with AC as chord and any inscribed angle ADC on the circumference. By reflecting the circle in AC, any inscribed angle ABC as shown in the new circle, would be equal to angle ADC. (In other words, ABCD would in general be an angle quad). If AB=AD is now given (property of skew kite), it follows that $\angle ACD = \angle ACB$ since equal chords in equal circles subtend equal inscribed angles on the circumference. Therefore triangles ABC and ADC are congruent $(\angle,\angle,s)$ from which follows that BC=DC.

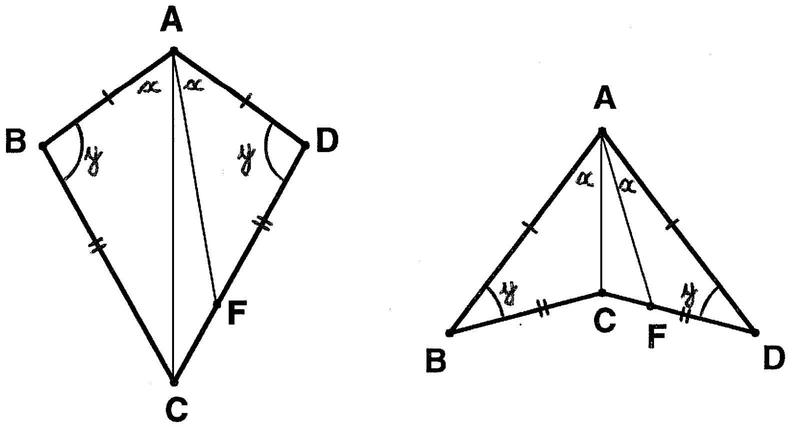


Figure 2.20

Alternatively, we could use *reductio ad absurdum.* For example, consider Figure 2.20. Assume BC ≠ DC and DC > BC. Mark F on DC so that DF = BC and connect A with F and C. Therefore triangles ABC and ADF are congruent $(s,\angle,s)$. This implies AF = AC. Furthermore, $\angle AFC = x+y$ (exterior angle) and therefore also $\angle ACF = x+y$. But $\angle ACB = 180°-x-y$ (sum of angles in ΔABC). Therefore $\angle BCD = \angle ACB + \angle ACF = 180°-x-y+x+y = 180°$ which is contradictory to the given assumption that C is a vertex of quadrilateral ABCD. Similarly can be shown that if BC > DC we obtain a contradiction. This means the assumption that BC ≠ DC is false, and therefore BC = DC.

(d) Yes. As before, we need only consider convex and concave cases. Consider Figure 2.21. We shall again use *reductio ad absurdum.* Assume BO ≠ OD and BO > OD. Extend OD to D' so that OD' = BO. Connect D' with A and C. Then triangles ABO and AD'O, as well as CBO and CD'O, are congruent $(s,\angle,s)$. This implies that triangles ABC and AD'C are congruent (s,s,s) and therefore $\angle ABC = \angle AD'C$. But $\angle ABC = \angle ADC$ is given, therefore $\angle AD'C = \angle ADC$, which is impossible, unless D' and D coincide. Similarly, if BO < OD we obtain a contradiction.

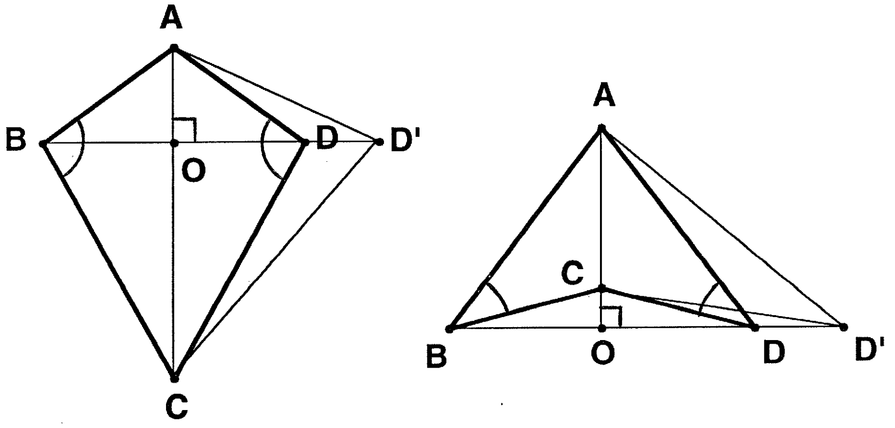


Figure 2.21

(e) Yes. It is only necessary to consider convex and concave cases, since a crossed skew-perpendicular quad is impossible. For example, suppose we have two adjacent sides, say, AB and AD equal as shown in Figure 2.22. Then for the diagonal AC to be perpendicular to diagonal BD, C now clearly must be chosen somewhere on the line of symmetry of triangle ABD, which implies only convex or concave possibilities for ABCD. (The proof for the convex and concave cases is left to the reader).

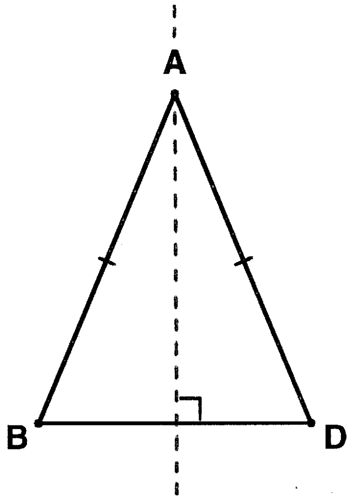


Figure 2.22

(f) One pair of opposite angles equal and one diagonal bisected by the other is not sufficient for a kite, as we can for example construct parallelograms from it as shown in Figure 2.23.

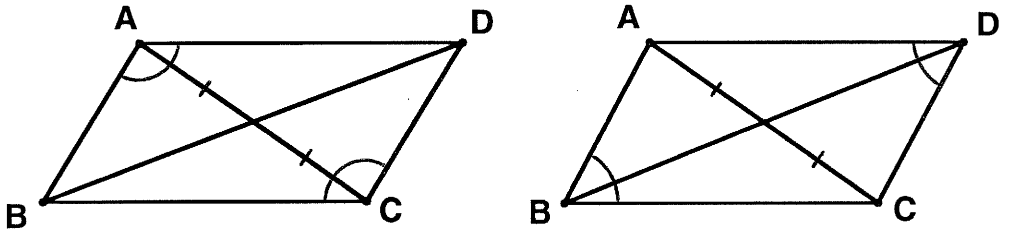


Figure 2.23

12. Consider the first figure in Figure 2.24. Let us agree to use the notation (ABC) to represent the area of triangle ABC. The area of the convex perpendicular quad can then be expressed as:

$$(ABC) + (ADC) = \frac{1}{2}AC.BO + \frac{1}{2}AC.OD = \frac{1}{2}AC(BO + OD) = \frac{1}{2}AC.BD.$$

In other words, its area is half the product of its diagonals. Similarly the area of concave perpendicular quadrilaterals (see Figure 2.24b) can be expressed as:

$$(ABC) - (ADC) = \frac{1}{2}AC.BO - \frac{1}{2}AC.OD = \frac{1}{2}AC(BO - OD) = \frac{1}{2}AC.BD$$

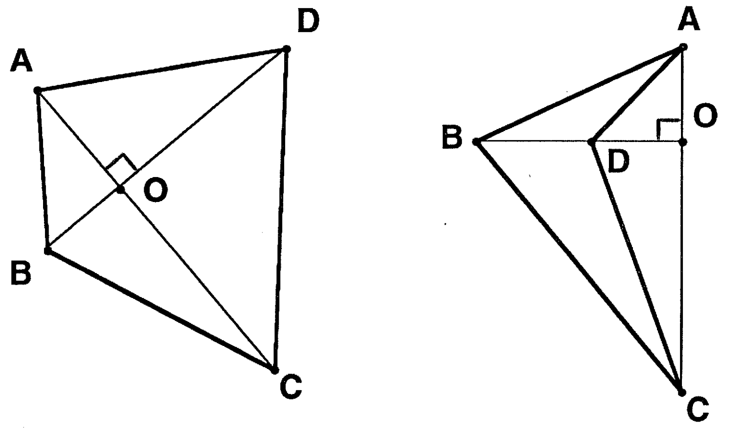


Figure 2.24

But what about crossed perpendicular quads? In order to extend the result to crossed perpendicular quads we must first consider what we mean by the *"area"* of a crossed quadrilateral. Let us now first carefully try and define a general area formula for convex and concave quadrilaterals. It is natural to define the area of a convex quadrilateral to be the sum of the areas of the two triangles into which it is decomposed by a diagonal. For example, diagonal AC decomposes the area as follows (see Figure 2.24a): (ABCD) = (ABC) + (CDA).

In order to make this formula also work for the concave case we obviously need to define (CDA)= -(ADC). In other words, we can regard the area of a triangle as being *positive* or *negative* according as its vertices are named in *counterclockwise* or *clockwise* order. For example:
(ABC) = (BCA) = (CBA) = -(CBA) = -(BAC) = -(ACB).

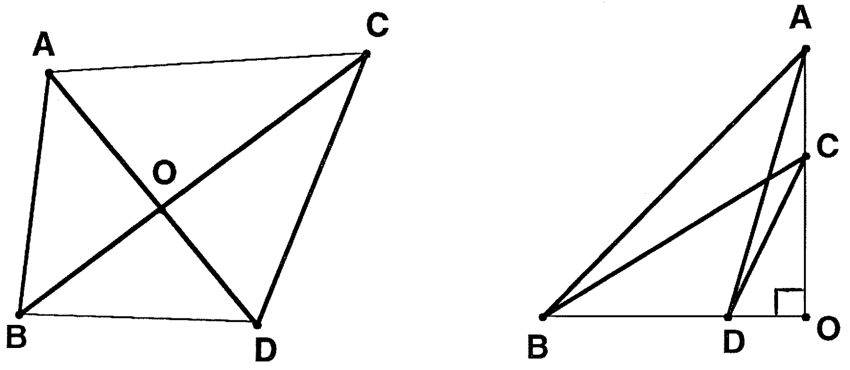


Figure 2.25

Applying the above formula and definition of area in a crossed quadrilateral (see first figure in Figure 2.25) we find that diagonal AC decomposes its area as follows:
(ABCD) = (ABC) + (CDA) = (ABC) - (ADC).

In other words, this formula forces us to regard the *"area"* of a crossed quadrilateral as the *difference* between the areas of the two small triangles ABO and ODC. [Note that diagonal BD similarly decomposes (ABCD) into (BCD) + (DAB) = -(DCB) + (DAB)]. An interesting consequence of this is that a crossed quadrilateral will have zero *"area"* if the areas of triangles ABO and ODC are equal.

We can now determine the area of a crossed perpendicular quad (see second figure in Figure 2.25) as follows:

(ABCD)= (ABC) - (ADC)= $\frac{1}{2}AC.BO-\frac{1}{2}AC.DO=\frac{1}{2}AC(BO-DO)=\frac{1}{2}AC.BD.$

In other words, its area is also half the product of its diagonals.

13. Consider Figure 2.26. Drop perpendiculars from A and C to BD as shown. Then triangles AHO and CJO are congruent $(\angle,\angle,s)$ and therefore AH = CJ. Therefore the areas of triangles DAB and BCD are equal (same base and heights).

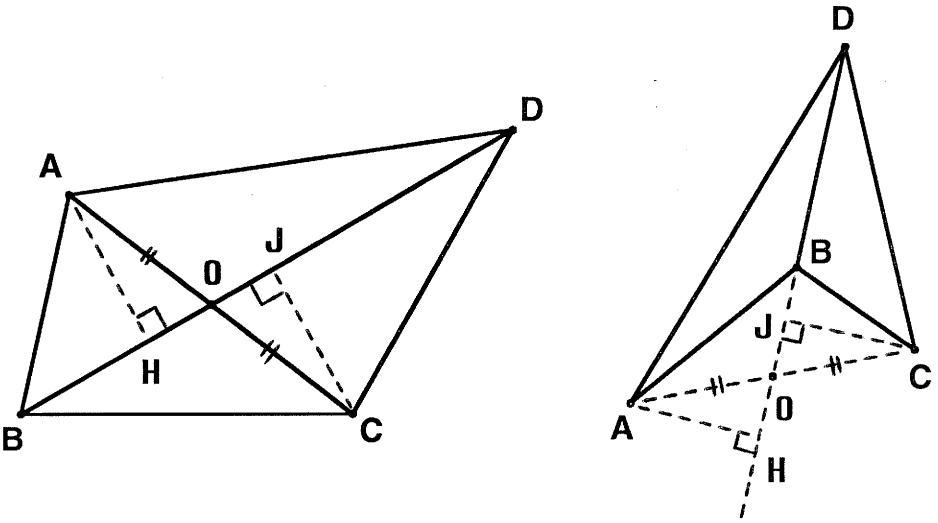


Figure 2.26

14. Yes. Consider Figure 2.27. In both cases, since equal chords subtend equal angles on the circumference of the circle, we have $\angle ABC=\angle DCB$ and $\angle BAD=\angle CDA$ in the corresponding arcs. In the convex case we have angles BAD and DCB supplementary, and therefore angles BAD and ABC are also supplementary which implies AD//BC. In the crossed case we have $\angle BAD=\angle DCB$, and therefore $\angle BAD=\angle ABC$ which implies AD//BC. It is now left as an exercise to the reader to verify in both cases that opposite sides AB and DC are equal.

[Note: We could choose to define an isosceles trapezium in such a way that it is confined to only convex cases. However, since crossed cases exhibit all the properties of the convex cases (e.g. one pair of opposite sides parallel, one pair of opposite sides equal, equal diagonals, cyclic, two pairs of adjacent angles equal, line of symmetry through one pair of opposite sides), we shall henceforth consider crossed cases of isosceles trapezia as well. Incidentally, which are the two pairs of equal adjacent angles in the crossed isosceles trapezium in Figure 2.27? (See *Solutions 2 (continued)*].

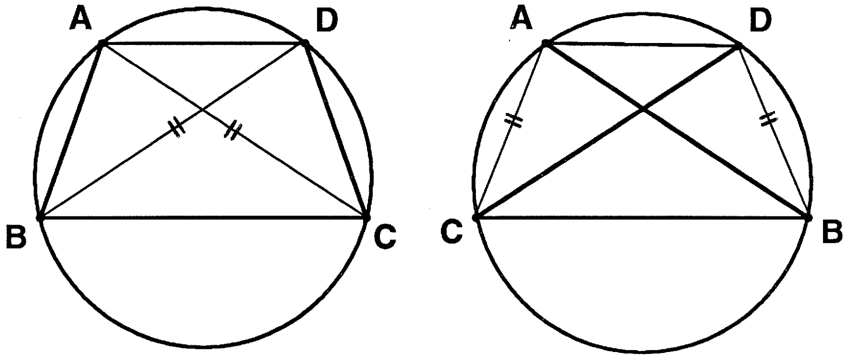


Figure 2.27

15. Yes. Consider Figure 2.28. Draw DE//AC with E on BC (extended in convex case). Then ACED is a parallelogram and DE=AC=DB. Therefore $\angle DEB = \angle DBE$ (triangle DBE is isosceles) and $\angle DEB = \angle ACB$ (corresponding angles with DE//AC) which implies $\angle ACB = \angle DBE$. In both cases we now have triangles ACB and DBC congruent $(s, \angle, s)$ and therefore opposite sides AB and DC are equal.

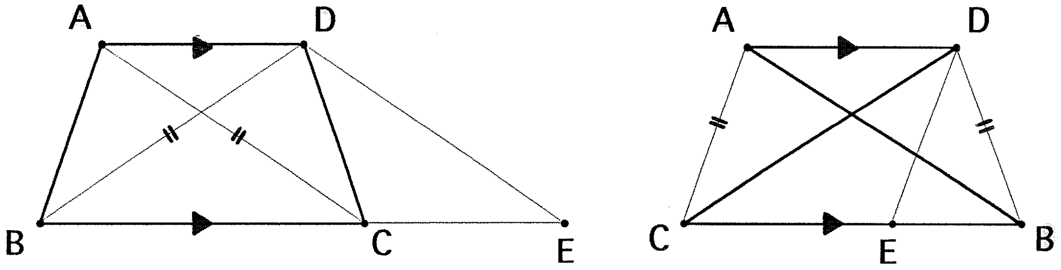


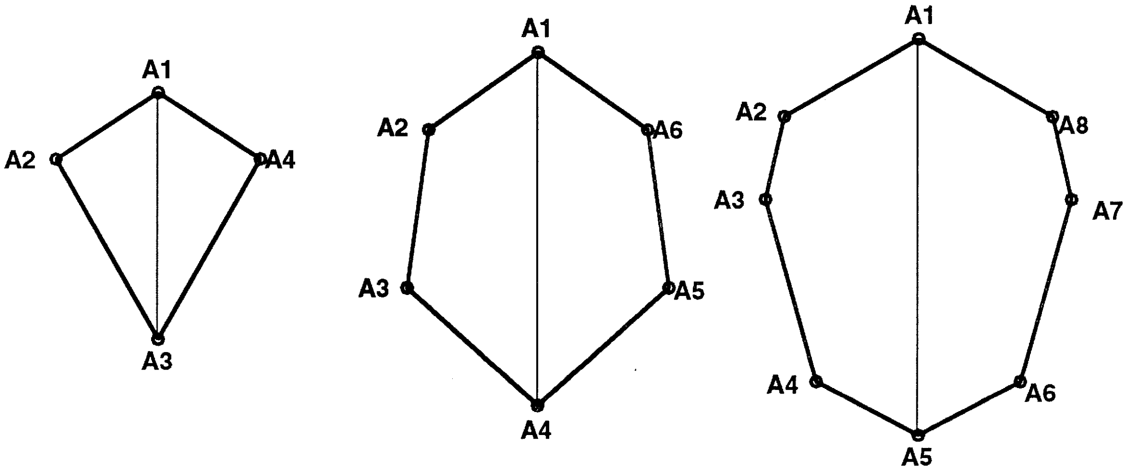
Figure 2.28

It is important to note a certain duality (or interchangeability) between opposite sides and diagonals for an isosceles trapezium. For a *convex* isosceles trapezium the diagonals intersect inside the figure, but for a *crossed* isosceles trapezium they take the place of opposite sides. Similarly, for a crossed isosceles trapezium the opposite sides now intersect inside the figure, in other words taking the place of the diagonals in the convex case. A crossed isosceles trapezium with equal diagonals is therefore equivalent to a convex isosceles trapezium with equal opposite sides, and similarly, a convex isosceles trapezium with equal diagonals is equivalent to a crossed isosceles trapezium with equal opposite sides. Using this duality we can now immediately formulate the following duals for problems 14 and 15 without proof (since their proofs would be the same as those for equal diagonals):

(a) A convex cyclic quadrilateral with equal *opposite sides* is an isosceles trapezium.

(b) A trapezium with the non-parallel pair of *opposite sides* equal is an isosceles trapezium.

16. The two sums of alternate sides in a kite $2n$-gon are equal. Consider Figure 2.29. From symmetry, we respectively have for a:
kite: $A_1A_2 + A_3A_4 = A_2A_3 + A_4A_1$
kite hexagon: $A_1A_2 + A_3A_4 + A_5A_6 = A_2A_3 + A_4A_5 + A_6A_1$
kite octagon: $A_1A_2 + A_3A_4 + A_5A_6 + A_7A_8 = A_2A_3 + A_4A_5 + A_6A_7 + A_8A_1$



Kite Kite hexagon Kite octagon

Figure 2.29

In general, we have from symmetry:
$A_1A_2 = A_{2n}A_1, A_2A_3 = A_{2n-1}A_{2n}, A_3A_4 = A_{2n-2}A_{2n-1}$,...etc. Therefore:
$A_1A_2 + A_3A_4 + \ldots + A_{2n-1}A_{2n} = A_2A_3 + \ldots + A_{2n-2}A_{2n-1} + A_{2n}A_1$.

Note that although a kite itself cannot be crossed, a kite hexagon, kite octagon, etc. may be crossed, but due to symmetry the above proof still applies. Now consider the converse: does a $2n$-gon with its two sums of alternate sides equal, necessarily have an axis of symmetry through one pair of opposite angles? (See *Solutions 2 (continued)*).

17. Consider Figure 2.30 for generalizations to quadrilaterals. Consider Figure 2.31 for three possible generalizations to $2n$-gons, namely, $2n$-gons with opposite sides *parallel* (parallel-$2n$-gons) or *equal* (equal -$2n$-gons), as well as $2n$-gons with opposite sides *parallel and equal* (parallelo -$2n$-gons). Note that a parallel-$2n$-gon does not necessarily have equal opposite sides, nor does an equal-$2n$-gon necessarily have opposite sides parallel.

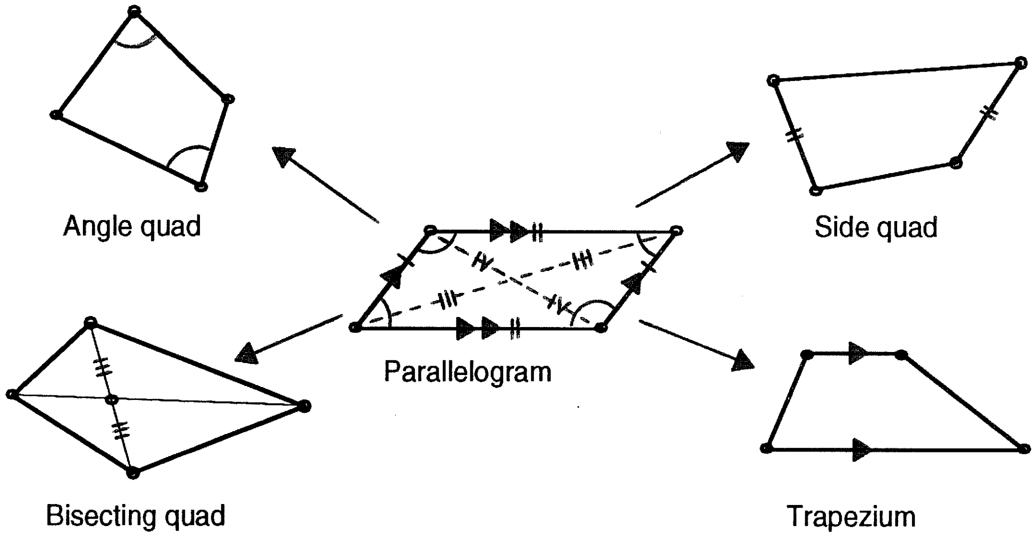


Figure 2.30

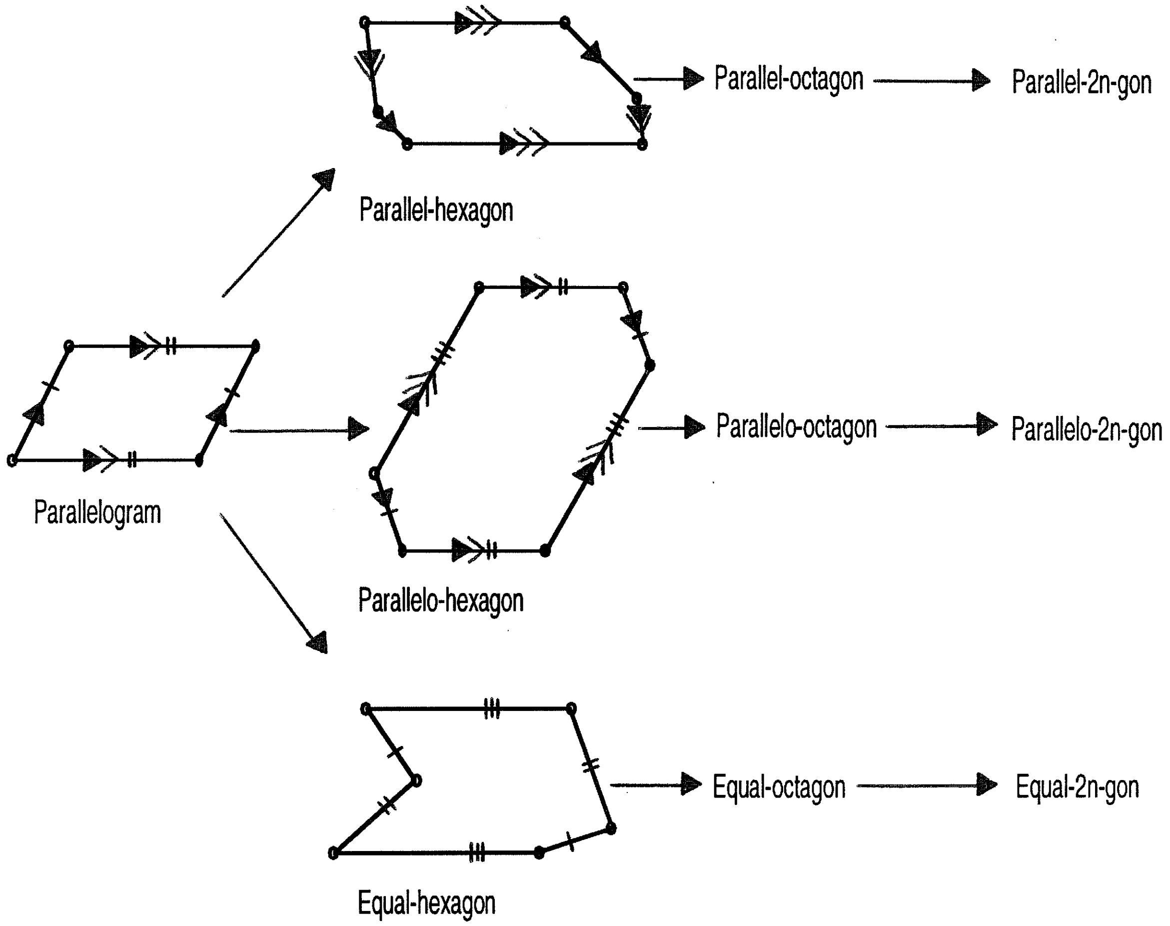


Figure 2.31

18. No, it is not true for a concave quadrilateral as shown by the counter-example in Figure 2.32 (where AB = CD and AO/OC = DO/OB = 9/3 = 3).

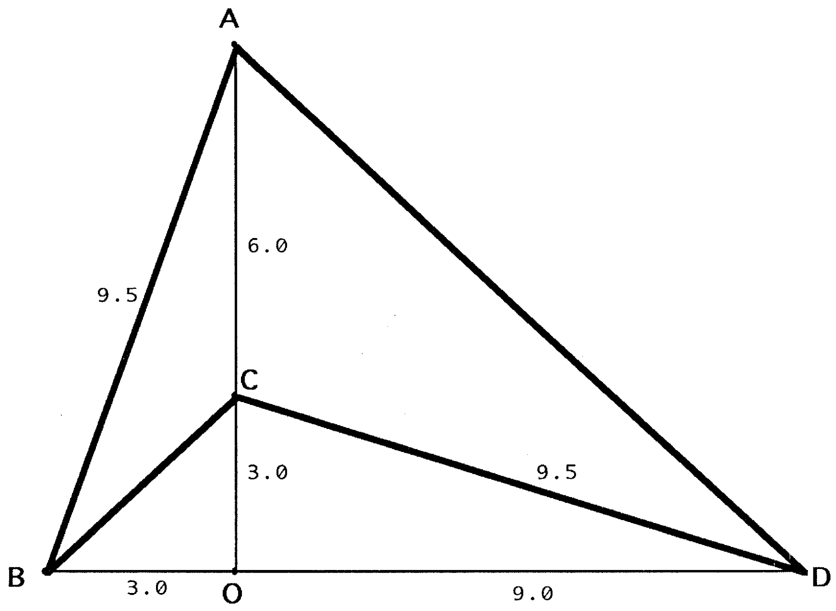


Figure 2.32

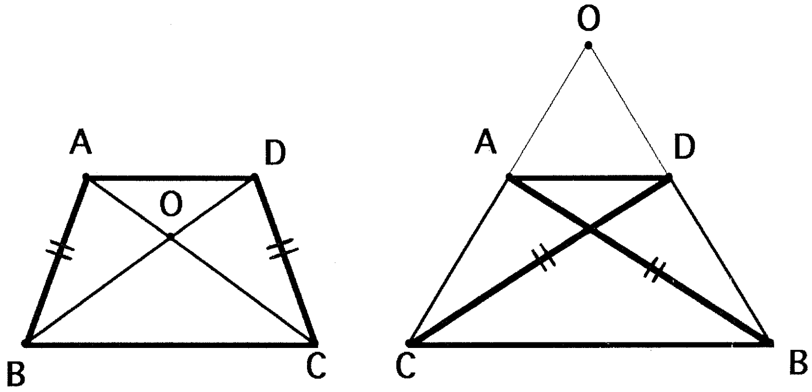


Figure 2.33

However, it is true for convex and crossed quadrilaterals as shown in Figure 2.33. In the first case, since $\angle AOD = \angle COB$ and the sides around these two angles are in ratio, we have that triangles AOD and COB are similar. This implies $\angle OCB = \angle OAD$ and therefore AD//BC. In the second case it follows that AD//BC from the well known result that if two points divide two sides of a triangle in ratio, then the line segment connecting them is parallel to the third.

Solutions 2 (continued)

2. (j) Although the given argument is quite valid for the figure shown, it is based on the *faulty assumption* that X and Y will always fall on AD and CB. This is unfortunately not always the case as shown by the counter-example in Figure 2.34 where X falls outside on AD extended and the above argument is invalid. This shows how important it sometimes is to empirically check one's deductive arguments for faulty assumptions by accurate construction and measurement (also see next chapter).

So this condition is necessary but **not** sufficient.

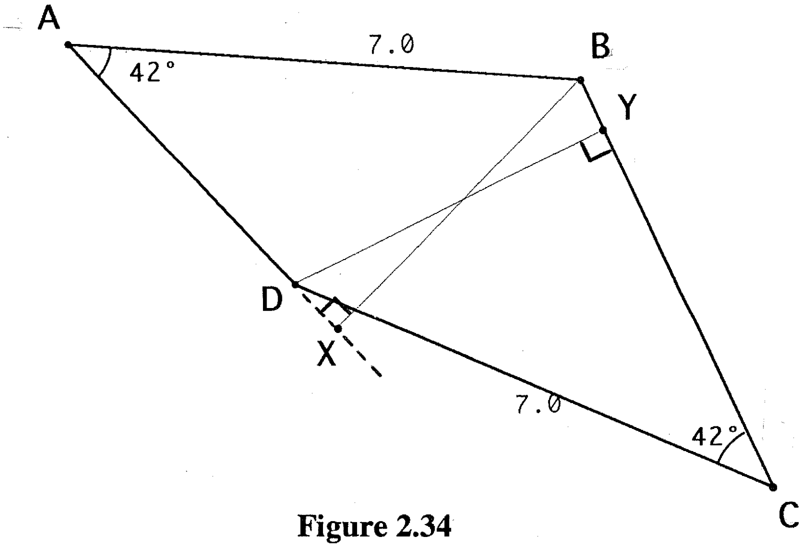


Figure 2.34

7. The sum of the interior angles of a crossed quadrilateral is 720°. Consider Figure 2.10(b) where we have:
$\angle BAD + \angle ABC = acute\angle BCD + acute\angle CDA$; therefore:
$\angle BAD + \angle ABC + (360° - \angle BCD) + (360° - \angle CDA) = 720°$.

14. Since A, B, C and D are labelled in a clockwise fashion (see solution 7) the two pairs of equal angles are $\angle ABC$ and $\angle DCB$, and $360° - \angle BAD$ and $360° - \angle CDA$ as shown in Figure 2.35.

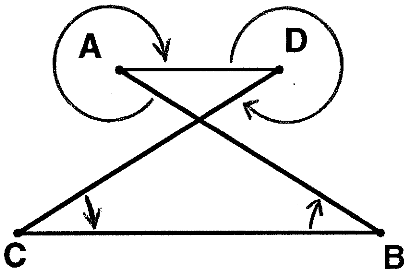


Figure 2.35

16. No, it is only true for quadrilaterals. Consider the counter-example shown in Figure 2.36.

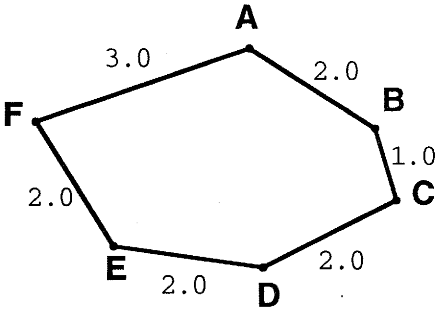


Figure 2.36

Solutions 3

1. Since equilateral triangles are all similar and also isosceles, one could from a hierarchical perspective try and generalize to *similar* triangles or *similar isosceles* triangles on the sides of the base triangle. Investigate further. (See *Solutions 3(continued)*).

2. See Chapter 5.

3. Yes, it is true and can be formulated as follows: "If three cevians AE, BF, CD satisfy $\frac{BE}{EC} \times \frac{CF}{FA} \times \frac{AD}{DB} = 1$, then they are concurrent."

   Proof

   Suppose that the cevians CD and BF meet at O as before (see Figure 32), and that the third cevian through this point is AE'. Then by the theorem of Ceva proved in the text, $\frac{BE'}{E'C} \times \frac{CF}{FA} \times \frac{AD}{DB} = 1.$

   But we are given that $\frac{BE}{EC} \times \frac{CF}{FA} \times \frac{AD}{DB} = 1.$

   Hence BE'/E'C = BE/EC and E' coincides with E, and we have proved that AE, CD and BF are concurrent.

4. (a) A circum quad obviously cannot be crossed since it would be impossible to construct a circle which touches all four its sides. However, a circumquad can be *concave* if we agree to allow the circle to touch the *extensions* of the sides as shown in Figure 3.1. Since CQ=CR=c, BP=BR=b and DS=DQ=d (tangents from the same points) we have AB + CD = $a+b+d-c$ = BC + AD; therefore the two sums of the pairs of opposite sides remain equal.

   (b) The converse can be formulated as follows: "Any quadrilateral with the two sums of its opposite sides equal, is a circum quad."

   Yes, it is true for *any* quadrilateral. Checking for the possibility of counter-examples in regard to crossed quadrilaterals the reader may have found that a crossed quadrilateral cannot have the sums of its opposite sides equal. In fact, construction and measurement shows that for any crossed quadrilateral ABCD as shown in Figure 3.2, we have AB + CD > BC + AD. In other words, the sum of the two intersecting opposite sides is always greater than the sum of the other two sides.

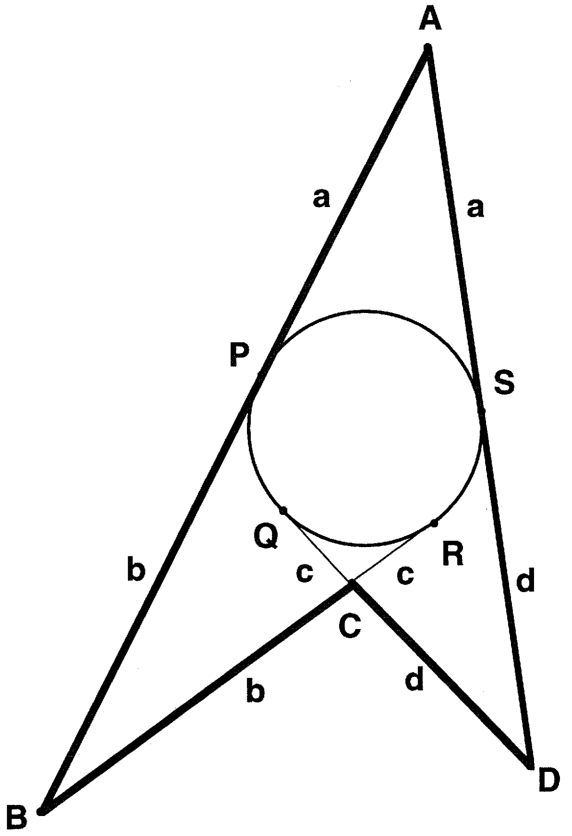


Figure 3.1

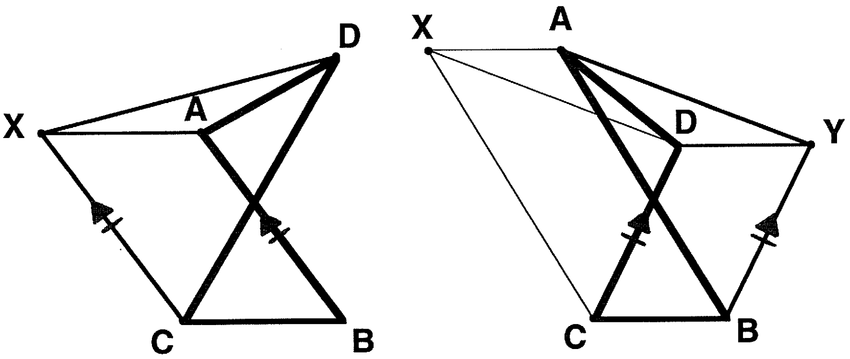


Figure 3.2

This can be proved as follows. In the first case translate BA to CX so that XCBA is a parallelogram. Connect X with D. Now in triangles XCD and XAD, we have XC + CD > XA + AD, and since XC=AB and XA=BC from the construction, we obtain the required result. In the second case, where A does not fall inside triangle XCD, we simply translate CD to BY as indicated and the proof follows as before in triangles ABY and ADY.

Note that this proof relies on the *lemma* that if two triangles have the same base and the one is contained by the other, then the sum of the remaining two sides of the larger one is greater than the sum of the corresponding sides of the smaller one. Consider Figure 3.3. Produce AO to meet the line CB at E.Then since a straight line is the shortest distance between two points we have CA + CE > AO + OE and BE + OE > BO. Adding these two inequalities we have CA+ CE + BE + OE > AO + OE + BO. Substitute for CE + BE its equal CB, and subtract OE from both sides to obtain the desired result CA + CB > OA + OB. (Note that the result is trivially true if O lies on AB).

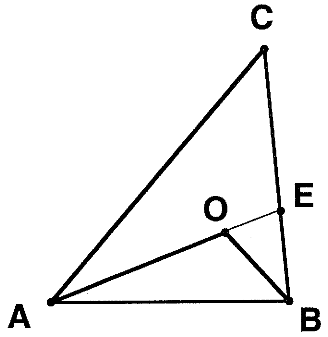


Figure 3.3

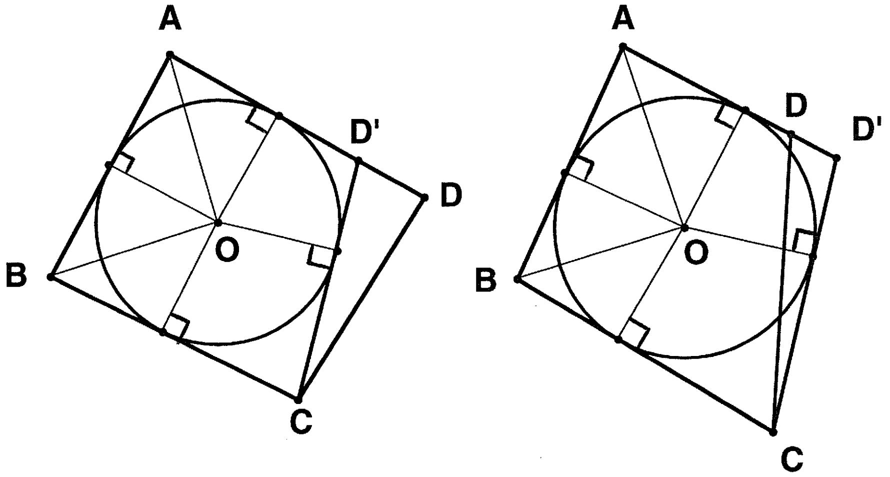


Figure 3.4

Proof of the converse

Construct the angle bisectors of angles A and B, and from their point of intersection O drop perpendiculars to sides AB, BC and AD. From O as center, construct a circle through these points so that sides AB, BC and AD are tangent to it (Figure 3.4). Assuming CD is not tangent to this circle, construct a tangent CD' as shown with D' on AD.

We now have AB + CD' = BC + AD', but we are given that AB + CD = BC + AD and therefore CD'- CD = AD'- AD or CD = CD'- AD' + AD. However, in the first figure AD = AD' + D'D and therefore CD = CD'- AD' + (AD' + D'D) = CD' + D'D which is impossible unless D'D = 0 and D' coincides with D. Similarly, in the second figure it follows that CD' = CD + DD' which is impossible unless DD' = 0 and D' coincides with D. This then completes the proof that CD is also tangent to the circle 0.

In a similar fashion the concave case can be proved, and is left as an exercise to the reader.

5. See Chapter 6.

6. No, the conjecture is false. For example, by first constructing AB = 6 cm, then AD = 4 cm (at some acute angle with AD), DC = 2cm and BC = 4,5cm, we obtain two distinct quadrilaterals, one concave and one convex, as shown in Figure 3.5. It is left to the reader to verify by measurement that the diagonals $EG_1 \neq EG_2$ and $HF_1 \neq HF_2$. (Note that another concave quadrilateral is possible by the reflection of triangle AC_1D in the line AC_1).

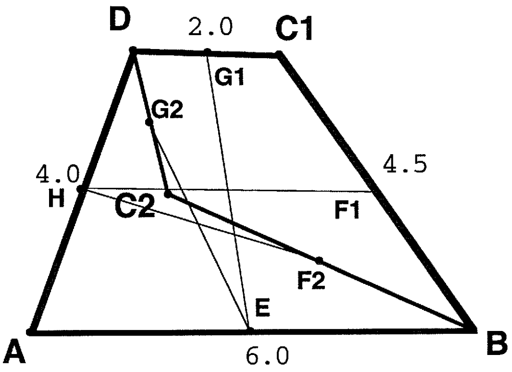


Figure 3.5

But might not be the conjecture true if we restrict it to say, convex quadrilaterals with given sides? Unfortunately this is also false. For example, for AB = 14,2 cm, BC = 13,1 cm, CD = 6,0 cm and DA = 9,2 cm, we find that the lengths EG and HF vary gradually as $\angle DAB$ varies. In particular, if $\angle DAB$ is increased from 57° to 97°, EG and HF increase gradually, respectively from 9,9 cm and 9,4 cm, to 10,6 cm and 10,1 cm. (Readers may want to verify this by calculation and also that EG and HF later on begin to decrease). Indeed, the variation is very small and slow and accounts for the apparent circular motion of G.

But might the conjecture not be true for certain special quadrilaterals? Investigate. (See *Solutions 3 (continued)*).

7. The triangle GHI is equilateral and the three circumcircles intersect in a common point, Can you explain why this result is true? Is there a relationship between it and the result referred to in problem no.1? Can you generalize? (See *Solutions 3 (continued)*).

8. Triangle XYX is equilateral. This result is known as Morley's theorem after Frank Morley who discovered it in 1904. The following proof is adapted from Johnson (1960:253 - 254). An alternative proof is given in Coxeter & Greitzer (1967:47-49).

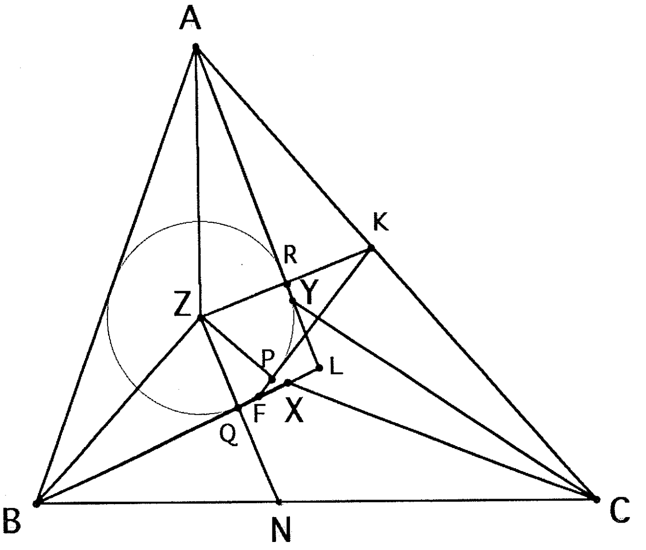


Figure 3.6

Extend AY and BX to meet at L. Draw the incircle of triangle ABL, whose center is obviously Z. Let Q and R denote its points of contact on LB and LA respectively, and let ZR extended meet AC at K, and $\overrightarrow{ZQ}$ meet BC at N; let the tangent from K to the circle touch it at P, and meet BL at F.

Now we have triangles ARZ and ARK congruent ($\angle,\angle,s$) and therefore ZR = RK and ZP = $\frac{1}{2}$ZK. From the trigonometry of right triangle ZPK it then follows that $\angle PZK = 60°$ and $\angle ZKP = 30°$. Since quadrilateral ZQLR is cyclic, we have $\angle QZR = 180° - \angle QLR$. But $\angle QLR = 180° - \frac{2}{3}\angle BAL - \frac{2}{3}\angle ABL = 60° + \frac{2}{3}\angle ACB$ since $\frac{2}{3}\angle BAL + \frac{2}{3}\angle ABL = 120° - \frac{2}{3}\angle ACB$. Therefore:
$\angle QZR = 120° - \frac{2}{3}\angle ACB$.

Furthermore, since BL is also a perpendicular bisector of NZ, we have $\angle FNQ = \angle FZQ$, and since XQZP is a kite,
$\angle FNQ = \frac{1}{2}\angle QZP = \frac{1}{2}(\angle QZR - 60°) = 30° - \frac{1}{3}\angle ACB$.
Since ZN = ZK, $\angle ZNK = \angle ZKN = 90° - \frac{1}{2}\angle QZR = \frac{1}{2}\angle QLR = 30° + \frac{1}{3}\angle ACB$. But $\angle ZKN = \angle ZKF + \angle FKN = 30° + \angle FKN$ and $\angle ZNK = \angle FNZ + \angle FNK = 30° - \frac{1}{3}\angle ACB + \angle FNK$, and therefore $\angle FKN = \frac{1}{3}\angle ACB$ and $\angle FNK = \frac{2}{3}\angle ACB$. Therefore, $\angle NFK = 180° - \angle FKN - \angle FNK = 180° - \angle ACB$. Therefore F, K, C and N are concyclic and $\angle FCN = \angle FKN = \frac{1}{3}\angle ACB$. But $\angle XCN = \frac{1}{3}\angle ACB$ which implies $\angle FCN = \angle XCN$. Since F and X lie on the same line BL, this would only be possible if F coincides with X, and therefore the tangent to the circle from K passes through X.

Similarly the tangent from N to the same circle touches it at P' and passes through Y. Now the figure ZNK is symmetrical, and NY and KX are symmetrically placed; hence it is easy to see that triangle ZXY is isosceles. Furthermore, arc PX = arc XQ = arc YR = arc YP' and therefore $\angle XZP = \angle P'ZY = \angle YZR$. But $\angle PZR = 60° = \angle PZP' + \angle P'ZY + \angle YZR$, and therefore $\angle XZY = \angle PZP' + \angle P'ZY + \angle XZP = 60°$, and triangle ZXY is equilateral.

9. The centres of the squares form a square. Can you explain why it is true? Can you generalize? (See *Solutions 3 (continued)*).

10. Quadrilateral $M_1M_2M_3M_4$ is a parallelogram. Can you explain why it is true? Is it also true if ABCD is a concave or crossed quadrilateral? Can you generalize? (See *Solutions 3 (continued)*).

11. Triangle PQR is equilateral. Can you explain why it is true? Can ABCD be a concave or crossed quadrilateral? If so, is it then still true? Can you generalize? (See Chapter 7).

12. Triangle PAB is equilateral. Can you explain why it is true? Can ABCD be a concave or crossed quadrilateral? If so, is it then still true? Can you generalize? (See Chapter 7).

13. Points P, Q and R are collinear (lie in a straight line). Can you explain why it is true? Can ABCD be a concave or crossed quadrilateral? If so, is it then still true? Can you generalize? (See Chapter 7).

15. (a) Pentagon = 180°; Star-Nonagon = 180°; Nonagon = 900°; Octagon = 360°.
(b) The angle sums remain constant. Although it is possible to prove this in each case by only using the exterior angle theorem and the sum of the interior angles of a triangle, we shall derive a general formula for the interior angles of such figures as follows. Consider any part of an eventually closing figure as shown in Figure 3.7. Imagine a turtle walking clockwise from A to B, turning through the angle x_1, moving along BC, turning through angle x_2, etc., until the figure closes and it is once again facing in the direction of AB. Clearly the sum of the turning angles must then be a multiple of 360°. Therefore, $\sum x_i = k \cdot 360°$... $k = 0;1;2;3;$ etc.

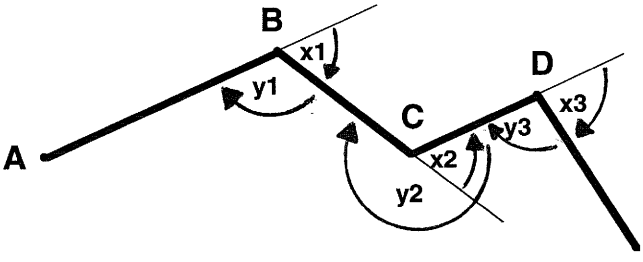


Figure 3.7

The sum of the interior angles is now simply the difference between $n \cdot 180°$ and the sum of the turning angles where n is the number of vertices. Therefore,

$$\begin{aligned} S = \sum y_i &= n \cdot 180° - \sum x_i \\ &= n \cdot 180° - k \cdot 360° \\ &= 180°(n - 2k) \end{aligned}$$

For simple closed polygons like triangles, convex or concave quadrilaterals, etc. $k = 1$ since one undergoes one full rotation walking around their perimeters (use a pen or pencil to rotate the one side on to the other, continuing around the perimeter and note the *total turning* of the pen or pencil). For example, for the pentagon $k = 2$, for the star-nonagon $k = 4$, for the other nonagon $k = 2$ and for the octagon $k = 3$. Substitution into the above formula therefore gives us the sums of the interior angles in each case.

It is interesting to note that a crossed quadrilateral has $k = 0$ (check if you don't believe it) and therefore according to the above formula the sum of its interior angles is 720°, and verifies what we discussed earlier in *Solutions 2 (continued)*, no. 7(b). For more examples, and a general classification of closed figures of this type, consult De Villiers (1989).

16. (a) This is not true for crossed cyclic quadrilaterals. It should therefore be reformulated as: all *convex* cyclic quadrilaterals have opposite angles supplementary. Similarly we can say: all *crossed* cyclic quadrilaterals have the sum of their opposite angles equal to 360°.

(b) This statement is true as it stands since a crossed quadrilateral cannot have opposite angles supplementary. However, it does not include the possibility of a crossed quadrilateral being cyclic. In order to do that we could reformulate it as follows: a quadrilateral with opposite angles supplementary, or adding up to 360°, is cyclic.

(c) Not true for any polygon as shown in 15(b) above. It should be reformulated to: the sum of the internal angles of any *simple closed* polygon is $(n-2)180°$.

17. By drawing for example a couple of parallelo-hexagons, it is easy to see that in general parallelo-$2n$-gons have point symmetry, opposite angles equal and diagonals bisecting one another at the point of symmetry.

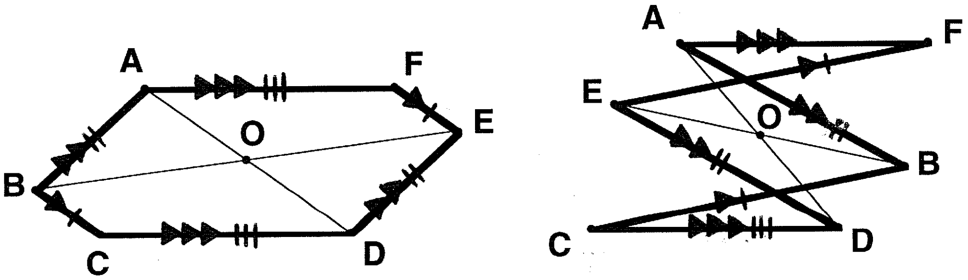


Figure 3.8

Consider Figure 3.8 showing a convex and a crossed parallelo-hexagon. Since

AB//=DE, we have that ABDE is a parallelogram with diagonals AD and BE bisecting each other in its point of symmetry O. Similarly BCEF is a parallelogram with diagonals BE and CF bisecting each other in say point O'. But since BE can only have one midpoint, O and O' must obviously coincide. Proceeding in this manner for any parallelo-2n-gon it is now easy to see that O would be a unique point of symmetry for the whole figure. Furthermore, since a half-turn maps the figure onto itself, we also have opposite angles equal.

Solutions 3 (continued)

1. Yes, it is indeed possible to generalize the result involving the Fermat point to similar triangles or similar isosceles triangles on the sides as follows:
"If similar triangles DBA, CBE and CFA are constructed outwardly on the sides of any ABC, then DC, EA and FB are concurrent " (see Figure 3.9).
"If similar isosceles triangles DBA, ECB and FAC are constructed outwardly on the sides of any ABC so that $\angle DAB = \angle DBA$, then DC, EA and FB are concurrent " (see Figure 3.10).

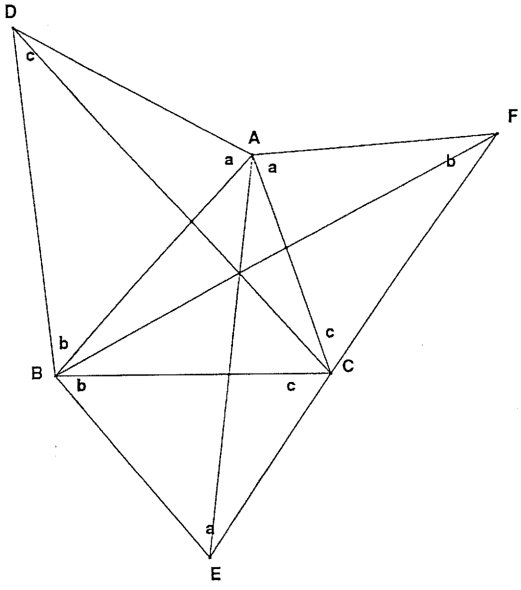


Figure 3.9

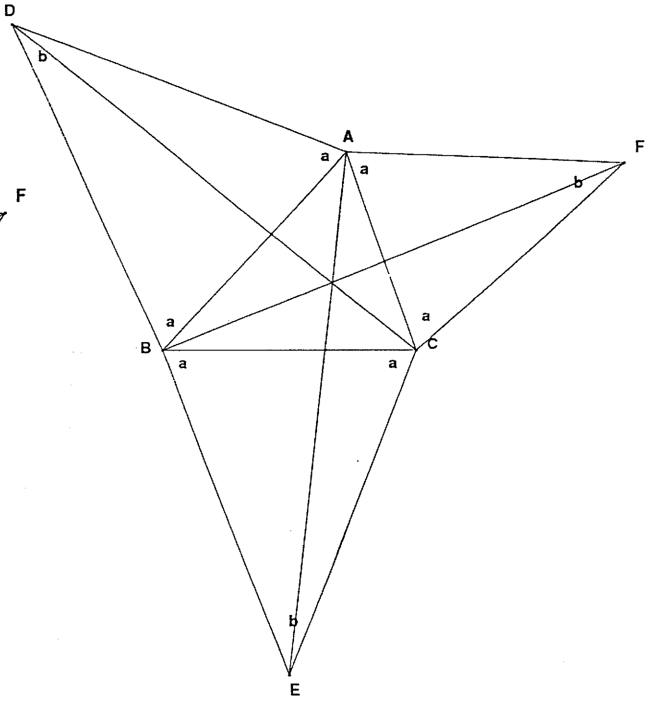


Figure 3.10

With regard to the first case note:

(a) if $a=\frac{1}{2}(\angle B+\angle C)$, $b=\frac{1}{2}(\angle A+\angle C)$ and $c=\frac{1}{2}(\angle A+\angle B)$, then DC, EA and FB are the angle bisectors of triangle ABC

(b) if triangles DBA, CBE and CFA are congruent, then DC, EA and BF are the altitudes of triangle ABC.

Also note with regard to the second case above:

(c) if $a = 0°$, then DC, EA and FB are the medians of triangle ABC.

These two results are therefore nice generalizations of the familiar concurrencies of the angle bisectors, altitudes and medians.

However, a further unifying generalization is possible, for example:
"If triangles DBA, ECB and FAC are constructed outwardly on the sides of any ΔABC so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle ACF$ then DC, EA and FB are concurrent."

To prove this result it is first necessary to prove the following lemma.

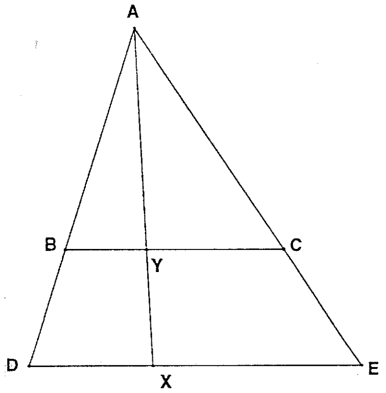


Figure 3.11

Lemma
Triangle ABC is given. Extend AB and AC to D and E respectively so that DE//BC. Choose any point Y on BC and extend AY to X on DE (see Figure 3.11). Then BY/YC = DX/XE.

Proof
Since triangles ABY and ADX are similar we have BY/YA = DX/XA and therefore BY = (YA/XA).DX ... (1). Similarly from the similarity of triangles ACY and AEX we have CY = (YA/XA).EX ... (2). Dividing (1) by (2), gives BY/YC = DX/XE, the desired result.

Proof of the Fermat generalization
Assume that the lines we want to prove concurrent intersect BC, CA and AB respectively at X, Y and Z. Extend AB to G and AC to H so that GEH//BC (see Figure 3.12). Label BE, EC, CF, FA, AD and DB respectively as s_1, s_2, s_3, s_4, s_5 and s_6. Then $\angle BGE = \angle ABC$ and $\angle BEG = b$.

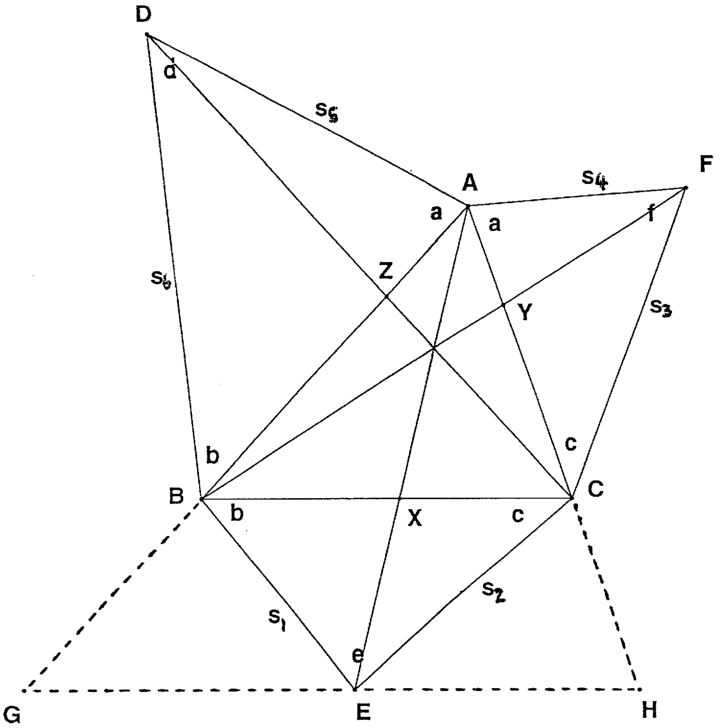


Figure 3.12

According to the sine rule:

$$\frac{GE}{\sin(\angle GBE)}=\frac{s_1}{\sin(\angle ABC)}$$

$$\frac{GE}{\sin(b+\angle ABC)}=\frac{s_1}{\sin(\angle ABC)}$$

$$GE=\frac{s_1\sin(b+\angle ABC)}{\sin(\angle ABC)}$$

Similarly we obtain

$$EH=\frac{s_2\sin(c+\angle ACB)}{\sin(\angle ACB)}.$$

According to the preceding Lemma therefore

$$\frac{BX}{XC}=\frac{GE}{EH}=\frac{s_1\sin(b+\angle ABC)}{\sin(\angle ABC)}\cdot\frac{\sin(\angle ACB)}{s_2\sin(c+\angle ACB)}.$$

In the same way we have

$$\frac{CY}{YA}=\frac{s_3\sin(c+\angle ACB)}{\sin(\angle ACB)}\cdot\frac{\sin(\angle CAB)}{s_4\sin(a+\angle CAB)}$$

$$\frac{AZ}{ZB}=\frac{s_5\sin(a+\angle CAB)}{\sin(\angle CAB)}\cdot\frac{\sin(\angle ABC)}{s_6\sin(b+\angle ABC)}$$

Therefore, $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{s_1}{s_2}\cdot\frac{s_3}{s_4}\cdot\frac{s_5}{s_6}$ ... (3)

Applying the sine rule to triangles ECB, FAC and DBA we obtain

$$\frac{s_1}{s_2}=\frac{\sin(c)}{\sin(b)};\frac{s_3}{s_4}=\frac{\sin(a)}{\sin(c)};\frac{s_5}{s_6}=\frac{\sin(b)}{\sin(a)}$$

By substitution into (3) therefore $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$ so that AX, BY and CZ are concurrent according to the converse of Ceva's theorem (see Problem 3). But then EA, FB and DC are also concurrent.

Although this result is not original, the author was not aware at the time of discovery that it had been published elsewhere before (See **Footnote** on p. 148). The heuristics of this discovery involved quasi-empirical exploration, as well as proof, and is described in De Villiers (1989 (a) & (b)). The first special case shown in Figure 3.9 was discovered and proved by Alwyn Olivier (Dept. of Mathematics Education, University of Stellenbosch). The second special case shown in Figure 3.10 was proved by Johan Meyer (Dept. of Mathematics, University of the Orange Free State) and the above proof is a generalization of his proof.

6. Interestingly, Noble's conjecture is true for parallelograms. Consider Figure 3.13 where ABCD is a parallelogram with sides of fixed length. Since H and F are the respective midpoints of equal opposite sides AD and BC, it follows that AH//=BF and therefore that ABFH is a parallelogram. Thus, HF=AB and therefore HF is also of fixed length. Similarly, EG=BC implying that EG is also fixed.

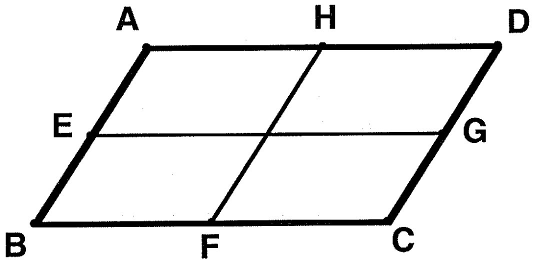


Figure 3.13

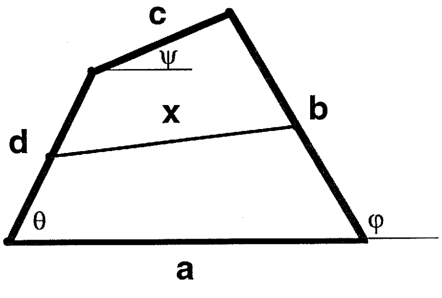


Figure 3.14

But what about the converse? Suppose instead that EG and HF are given with fixed lengths. Which different kinds of quadrilaterals could we get, or is it only parallelograms for which that is possible?

The author is indebted to John Webb, Dept. of Mathematics, University of Cape Town for the following proof showing that EG and HF are of fixed length only for parallelograms.

Consider Figure 3.14. Sides a, b, c and d of a convex quadrilateral are given with angles θ, φ and ψ as shown. The angles are variable, but only one (say φ) is independent, lying in some range [$\varphi_{\min}, \varphi_{\max}$]. Then
$c\cos\psi = a + b\cos\varphi - d\cos\theta$ ...(1)
$c\sin\psi = |b\sin\varphi - d\sin\theta|$ ...(2)

Squaring (1) and (2) and adding leads to
$2a(d\cos\theta - b\cos\varphi) + 2bd\cos(\theta-\varphi) = a^2 + b^2 - c^2 + d^2$ ...(3).

If the midpoints of the sides of lengths a and b are joined by a line of length x, we obtain in the same way:
$2a(\frac{d}{2}\cos\theta - \frac{b}{2}\cos\varphi) + 2\frac{b}{2}\frac{d}{2}\cos(\theta-\varphi) = a^2 + b^2 - x^2 + d^2$ ...(4).

Let's now assume that x is constant and independent of φ and θ. From (3) and (4) we can deduce that $\cos(\theta-\varphi)$ and $d\cos\theta - b\cos\varphi$ are constant (and hence ψ is constant). Let $\theta - \varphi = \alpha$ (constant). Then, for $\varphi \in [\varphi_{\min}, \varphi_{\max}]$:
$d\cos(\varphi+\alpha) - b\cos\varphi = k$ (constant).

Differentiate: $-d\sin(\varphi+\alpha) + b\sin\varphi = 0$ and again: $-d\cos(\varphi+\alpha) + b\cos\varphi = 0$. It now follows that $b = d$ and $\alpha = 0$. Hence the quadrilateral is a parallelogram.

7. This result is known as Napoleon's theorem. Although the famous French Emperor was greatly interested in geometry, it is not certain whether he was actually the original discoverer of this result. Comparing this result with Problem 1, it should be clear that the common point of intersection O of the three circumcircles is the same point as the Fermat point.

Proof

Construct circumcircles ADB and BEC to intersect in B and O (see Figure 3.15). Joining O with A, B and C, we see that $\angle BOC = 180° - \angle E (= 120°)$ and $\angle AOB = 180° - \angle D (= 120°)$ and so

$$\begin{aligned}\angle AOC &= 360° - (\angle BOC + \angle AOB)\\ &= 360° - (180° - \angle E + 180° - \angle D)\\ &= \angle E + \angle D ...(= 120°)\\ &= 180° - \angle F\end{aligned}$$

Therefore $\angle AOC$ and $\angle F$ are supplementary, and O lies on the circumcircle of ΔAFC. Now since GH, HI and GI are respectively perpendicular to the common chords OB, OC and OA, it follows that OKGJ, OKHL and OLIJ are all cyclic quadrilaterals. Therefore, $\angle I$ must be the supplement of $\angle AOC$, but from the aforementioned result we have $\angle F$ also the supplement of $\angle AOC$. Therefore, $\angle I = \angle F (= 60°)$ and in the same way it follows that $\angle G = \angle D (= 60°)$ and $\angle H = \angle E (= 60°)$.

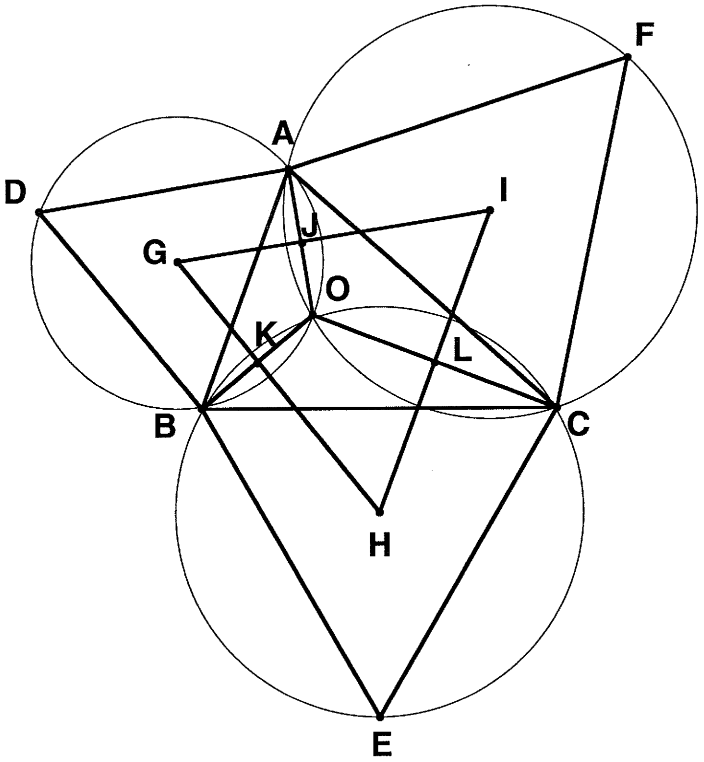


Figure 3.15

Looking back

Careful analysis of the above proof, shows that the pairs of angles I and F, G and D and H and E would remain equal if $\angle E + \angle D = 180° - \angle F$ or rather $\angle D + \angle E + \angle F = 180°$. In other words, we can immediately generalize Napoleon's theorem as follows:

"If triangles DBA, BEC and ACF are erected externally on the sides of any ΔABC so that $\angle D + \angle E + \angle F = 180°$, their circumcircles meet in a common point and their circumcenters G, H and I form a triangle $\angle G = \angle D$, $\angle H = \angle E$ and $\angle I = \angle F$" (see Figure 3.16).

The following two interesting special cases also follow directly from this generalization:

"If *similar* triangles DBA, BEC and ACF (or DBA, CBE and CFA) are erected externally on the sides of any ΔABC, their circumcenters G, H and I form a triangle similar to the three triangles."

The two cases are illustrated in Figure 3.17. This is therefore another good example of the power of proof, not only as a means of explanation, but also of further discovery.

It should furthermore be noted that the general Fermat point (the point of concurrency of CD, AE and BF) referred to in Problem 1, coincides with the common point of intersection of the three circumcircles shown in Figure 3.17b. In fact, this is the most general configuration in which this can occur. For example, consider Figure 3.12 where $\angle D = 180° - a - b$, $\angle E = 180° - b - c$ and $\angle F = 180° - a - c$. If we now add the further condition $\angle D + \angle E + \angle F = 180°$ to ensure that the circumcircles have a common point of intersection, it follows that $a + b + c = 180°$. Therefore, $c = 180° - a - b$ and equal to angle D. Similarly, $\angle E = a$ and $\angle F = b$.

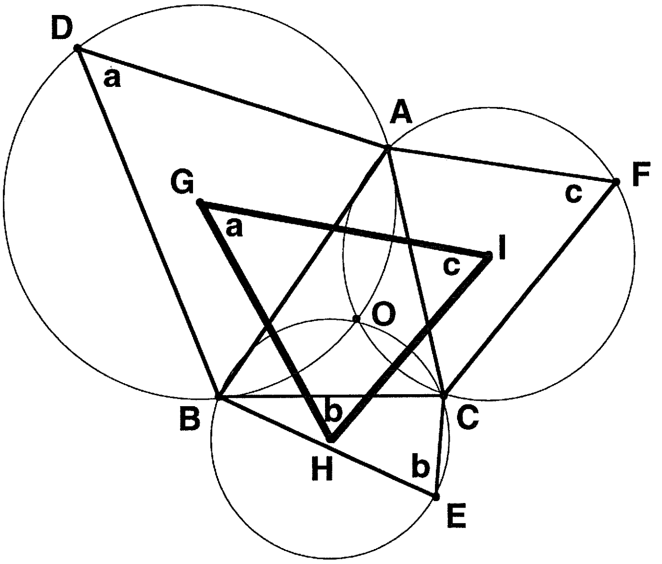


Figure 3.16

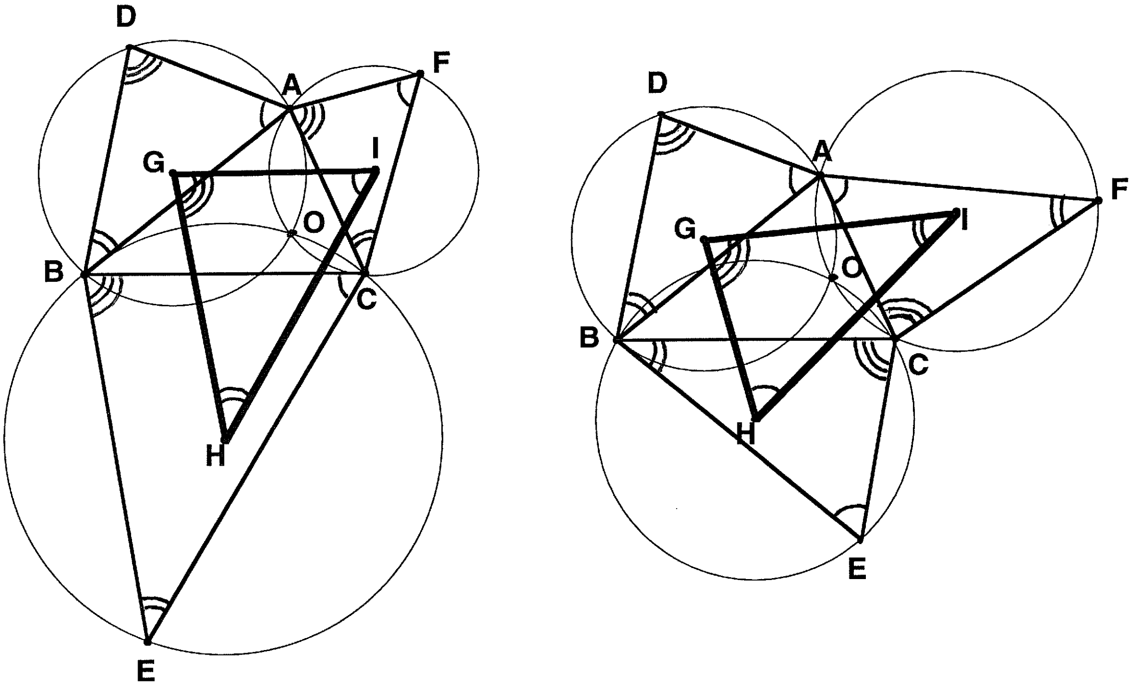


Figure 3.17

Lastly, the adventurous reader may also want to consider the following *"what-if"* questions for all the various cases:

(a) what happens if the triangles are all constructed *inward* on the sides of the base triangle?

(b) what happens if *squares* or other *quadrilaterals* are constructed on the sides of the base triangle?

(c) what happens if the base triangle is replaced by a *quadrilateral* or *pentagon*?

9. Consider Figure 3.18. A half-turn clearly maps the whole figure onto itself, and therefore EFGH is a parallelogram. Draw triangles EBF and EAH. Then $\angle EBF = \angle EBA + \angle B + \angle FBC = 90° + \angle B$. Also $\angle XAY = 360° - \angle XAB - \angle A - \angle YAD = 360° - 180° - \angle A$ and therefore $\angle EAH = 180° - \angle A + 90° = 270° - \angle A$.
But $\angle A = 180° - \angle B$ and therefore $\angle EAH = 270° - (180° - \angle B) = 90° + \angle B$. Therefore triangles EBF and EAH are congruent $(s, \angle, s)$ and EF=EH. Therefore EFGH is a rhombus. Furthermore, since an anti-clockwise rotation through 90° around E maps triangle EBF onto triangle EAH $(\angle BEA = 90°)$, we also have $\angle FEH = 90°$. But a rhombus with a right angle is a square.

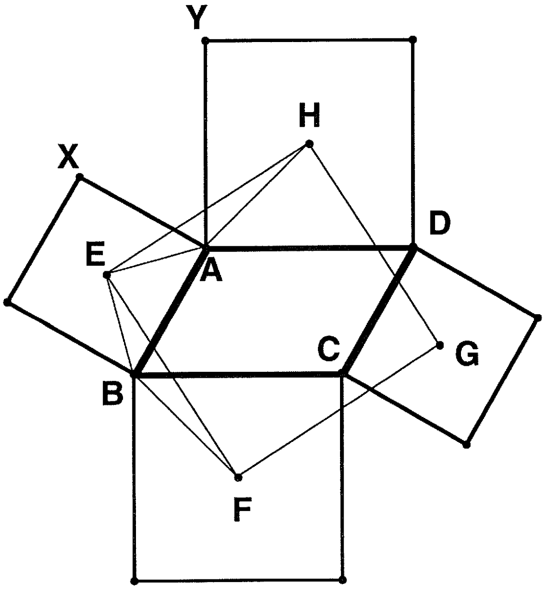


Figure 3.18

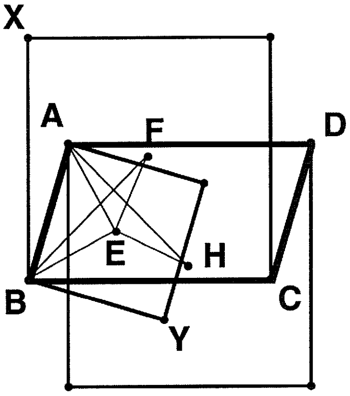


Figure 3.19

What happens if the squares are constructed *inward*? Interestingly, the result remains true as shown in Figure 3.19. However, here we have $\angle EAH = \angle A - 90°$. Furthermore, $\angle XBA = 90° - \angle B = \angle YBC$. Therefore $\angle XBY = 90° - \angle B + \angle B + 90° - \angle B = 180° - \angle B = \angle A$. But $\angle EBF = \angle XBY - \angle XBF - \angle YBE = \angle A - 90°$. Since AH=BF and EB=EA as before, we again have triangles EBF and EAH congruent $(s, \angle, s)$ and the rest of the proof follows similarly.

What if we had other *similar figures* on the sides? Interestingly, EFGH remains a parallelogram since the figure as a whole still has half-turn symmetry. Any similar figures on the sides can therefore be used with corresponding points connected as shown in Figure 3.20. In fact, all the figures do not have to be similar to each other. It is only necessary to have the figures on each pair of opposite sides similar and their corresponding points connected.

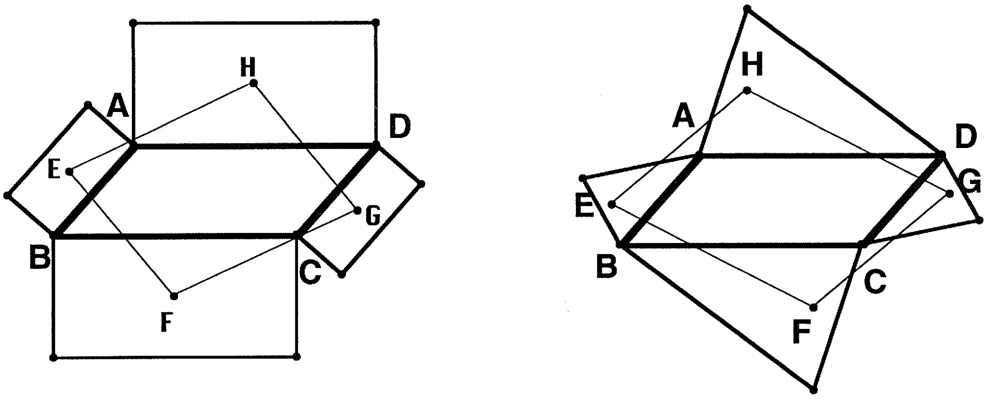


Figure 3.20

Another *"what-if"* question: what would happen if we constructed squares on the sides of *any quadrilateral*? This question and other further possible generalizations of this result will be addressed in *Questions & Problems 4*, no. 20.

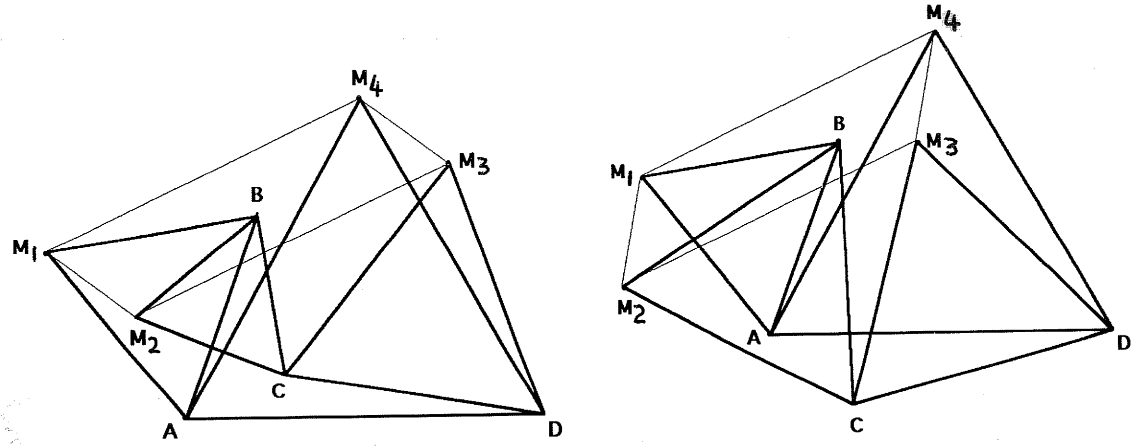


Figure 3.21

10. This result is not only true for convex quadrilaterals, but also for concave and crossed quadrilaterals as shown in Figure 3.21. Furthermore, we can generalize the result to the construction of *similar triangles* M_1BA, M_2BC, M_3DC and M_4DA as shown in Figure 3.22. The following proof is a generalization of the proof by Yaglom (1962:39, 94-95) for the special case for equilateral triangles.

Proof

The sum of the four spiral similarities $(k;z)$, $(1/k;z)$, $(k;z)$ and $(1/k;z)$ around centres M_1, M_2, M_3 and M_4, where the direction of the first and third rotations is opposite to that of the second and fourth, carries the vertex A of the quadrilateral onto itself. (Note that $k = M_1B / M_1A = M_2B / M_2C = M_3D / M_3C = M_4D / M_4A$ from the similarity of the triangles). But according to Lemma 1 (see further on) the sum of the two spiral similarities about M_1 and M_2 is a translation of M_1 by the segment M_1M_1' where M_1' is a vertex of the triangle $M_2M_1'M_1$, which is similar to the triangles M_1AB, M_2CB, etc. constructed on the sides of the quadrilateral ($M_2M_1 / M_2M_1' = k, \angle M_1M_2M_1' = z$ and the direction of rotation from M_2M_1 to M_2M_1' coincides with the direction of rotation from M_2B to M_2C).

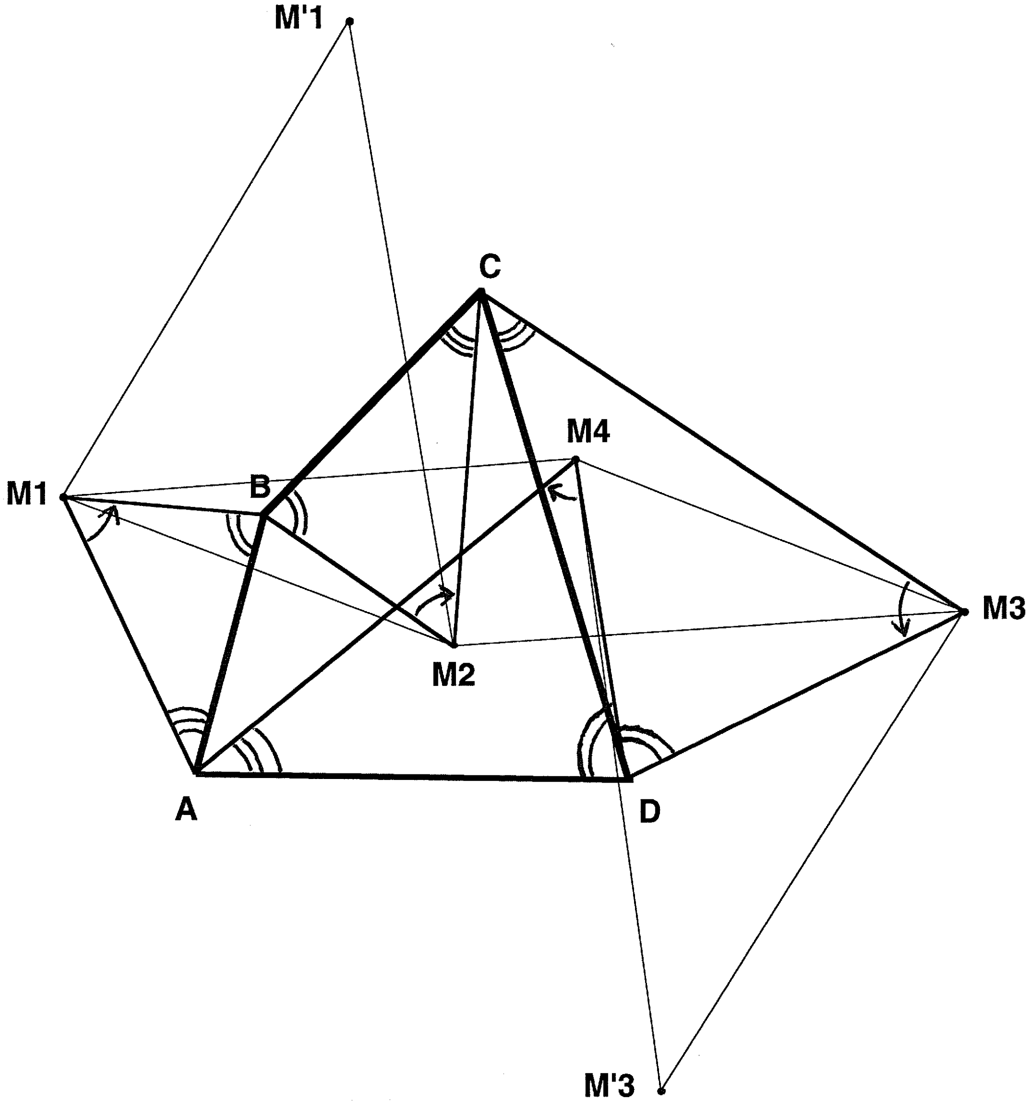


Figure 3.22

Similarly the sum of the two spiral similarities about M_3 and M_4 is a translation given by the segment M_3M_3' where triangle $M_4M_3'M_3$ is similar to triangles M_1AB, M_2CB, etc. and also to $M_2M_1'M_1$ (the direction of rotation from M_4M_3 to M_4M_3' is the same as the direction of rotation from M_4D to M_4A).

Thus the sum of the two translations given by the segments M_1M_1' and M_3M_3' carries the point A into itself. But if the sum of two translations leave even one point fixed, then this sum must be the identity transformation, that is, the two segments that determine the two translations must be equal, parallel and oppositely directed. But if $M_1M_1' = M_3M_3'$ then the similar triangles $M_2M_1'M_1$ and $M_4M_3'M_3$ are congruent, and therefore $M_1M_2 = M_3M_4$. Since the two congruent triangles $M_2M_1'M_1$ and $M_4M_3'M_3$ are furthermore so situated that M_1M_1' is parallel and oppositely directed to M_3M_3' it follows that M_1M_2 and M_3M_4 must also be parallel and oppositely directed, and therefore that the quadrilateral $M_1M_2M_3M_4$ is a parallelogram.

The other cases can be proved in a similar fashion and are left as exercises to the reader.

Note: A *spiral similarity (k,x)* is simply an *enlargement* or *reduction* from a particular point with a factor *k*, followed by a *rotation* around the same point through the angle *x*, or vice versa.

Lemma 1

The sum of two spiral similarities (k,α) and $(1/k,\beta)$ about centres O_1 and O_2 is equivalent to a translation if α and β are equal in size and opposite in direction.

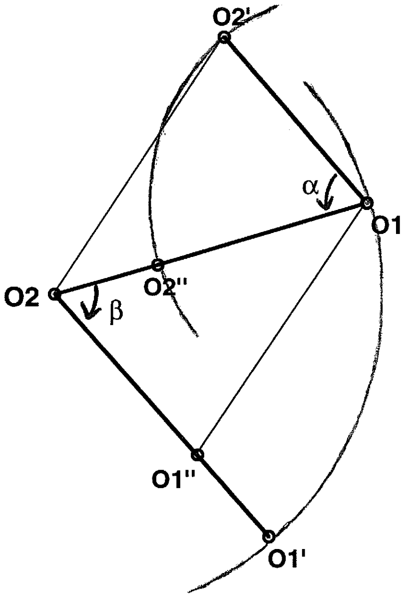


Figure 3.23

Proof

Consider Figure 3.23. The first spiral similarity leaves O_1 in place, and the second carries O_1 into O_1' by the rotation of β and then into O_1'' by the dilation with factor $1/k$. The sum of the two spiral similaries therefore carries the centre O_1 into a point O_1'' such that $O_1''O_2 = (O_1O_2)/k$.

The first spiral similarity carries a point O_2' into O_2'' by the rotation of α and then into O_2 by the dilation with factor k. (Therefore $O_2''O_1 = O_1O_2' = (O_1O_2)/k$). The second spiral similarity leaves O_2 in place. The sum of the two spiral similarities therefore carries a point O_2' into O_2.

It now follows that $O_2O_1''O_1O_2'$ is a parallelogram since $O_1''O_2 // = O_2'O_1$. Therefore the sum of the two spiral similarities is equivalent to a translation which carries O_2' and O_1, respectively along line segments $O_2'O_2$ and O_1O_1'', into O_2 and O_1''. (Also note that $O_2O_1 / O_2O_1'' = k$).

Footnote to Problem 1

The Fermat point is also called the Torricelli point. (Apparently the French and Italians have been arguing since the seventeenth century about who was "*first*").

The author has also since learned of an earlier and rather simpler proof of the unifying generalization given on p. 138 by A. R. Pargeter in **The Mathematical Gazette**, Vol. 47, no. 364, pp. 218-219, and an even earlier 1936 proof by N. Alliston in the **Mathematical Snack Bar** by W. Hoffer, pp. 13-14. (These earlier proofs, however, do not mention the interesting associated result given in Equation (3) on p. 140, namely, that the product of the given side ratios of the constructed triangles (or their reciprocals) is always equal to 1).

Solutions 4

1. (a) An *isosceles circumtrapezium;* in other words, an isosceles trapezium circumscribed around a circle. The duality between a right kite and an isosceles circumtrapezium is illustrated in Figure 4.1 where they are both inscribed and circumscribed around the same circles. (Is it possible to get concave right kites and crossed isosceles circumtrapezia?).

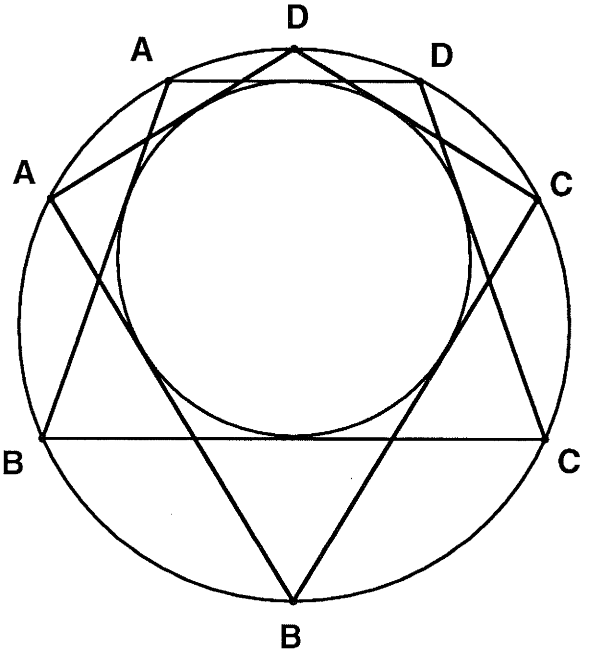


Figure 4.1

Furthermore, if we let the two equal sides of an isosceles circumtrapezium be represented by a, and the other two parallel sides by b and c, then we have $2a = b + c$ or $a = (b+c)/2$. Similarly, if we let the two equal angles of a right kite be represented by $\angle A$, and the other two angles by $\angle B$ and $\angle C$, then we have $2\angle A = \angle B + \angle C$ or $\angle A = (\angle B + \angle C)/2$. (In general, the former property is of course a property of any circum quad with (at least) one pair of equal opposite sides (*circum side quad*) while the latter is a property of any cyclic quad with (at least) one pair of equal opposite angles (*cyclic angle quad*) as shown in Figure 4.2).

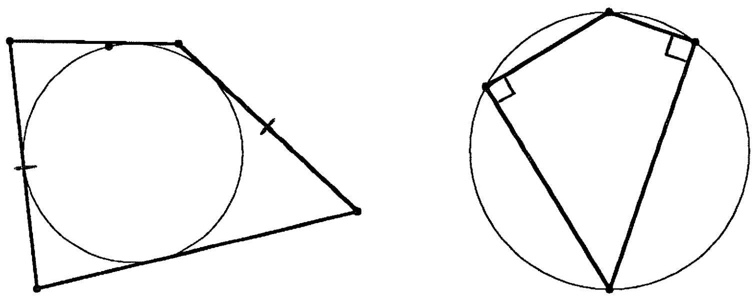


Figure 4.2

(b) A *triangular kite*; in other words, a kite with (at least) three equal angles (see Figure 4.3). Similar to the trilateral trapezia, the set of squares is the only subset. Can you find a dual property for a triangular kite to the property of a trilateral trapezium that its diagonals bisect a pair of equal adjacent angles as shown in *Questions and Problems 1*, no. 3? (See *Solutions 4 (continued)*).

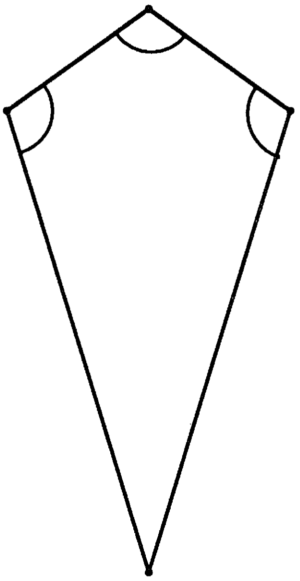


Figure 4.3

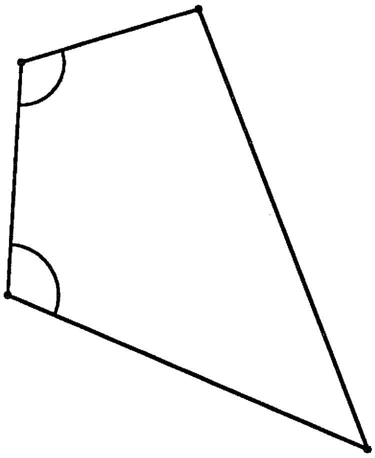


Figure 4.4

(c) A *skew isosceles trapezium*; in other words, a quadrilateral with (at least) two equal adjacent angles (see Figure 4.4). Also see no's. 3 and 9 later on.

(d) A *skew circum quad*; in other words, a circum quad with two equal adjacent angles (see Figure 4.5a). Can you find a dual property for skew circum quads to the property of skew cyclic quads that the angle, opposite the angle enclosed by the two equal sides, is bisected by a diagonal (see Figure 4.5b)? (See *Solutions 4 (continued)*).

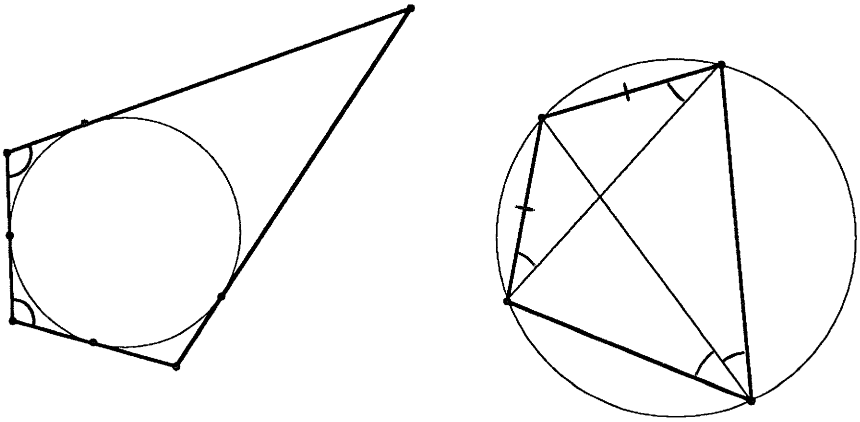


Figure 4.5

2. The angle and side quads are obviously dual, and it would appear as if the bisecting quad and trapezium are dual. However, there does not seem to be interesting duals

between their respective properties. For example, in a trapezium with sides AD // BC we have the sums of two pairs of adjacent angles equal, e.g. $\angle A + \angle B = \angle C + \angle D$ in Figure 4.6a, but it is not necessarily the case that for a bisecting quad with say BO = OD that the sums of two pairs of adjacent sides are equal (see Figure 4.6b).

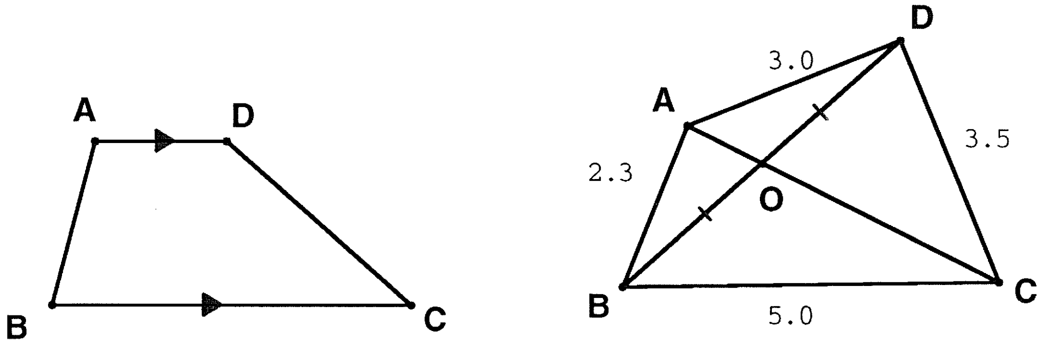


Figure 4.6

The duality between them however becomes apparent when we consider some of their intersections with other quadrilaterals. As shown earlier in *Solutions 2*, no. 15, a trapezium with equal diagonals is an isosceles trapezium, while a bisecting quad with perpendicular diagonals is a kite (see *Solutions 2*, no. 11a) . We can now use this duality to form the following conjectures.

Conjecture 1

For example, in *Solutions 2*, no. 11(d) we proved that an *angle quad* with *perpendicular diagonals* is a *kite.* This suggests that a *side quad* with *equal diagonals* would be an *isosceles trapezium.*

Conjecture 2

Similarly, since a *side quad* that is *cyclic* is an *isosceles trapezium* (see *Solutions 2*, no. 15), we have the conjecture that an *angle quad cicumscribed* around a circle would be a *kite.*

Conjecture 3

Similarly, since a *trapezium* that is *cyclic* is an *isosceles trapezium* (see Chapter 1), we have the conjecture that a *bisecting quad circumscribed* around a circle would be a *kite.*

Investigate these three conjectures. If true, provide proofs. If not, provide counter-examples. (See *Solutions 4 (continued)*).

(3) We can generalize the concept isosceles trapezium to at least five different quadrilateral generalizations as shown in Figure 4.7.

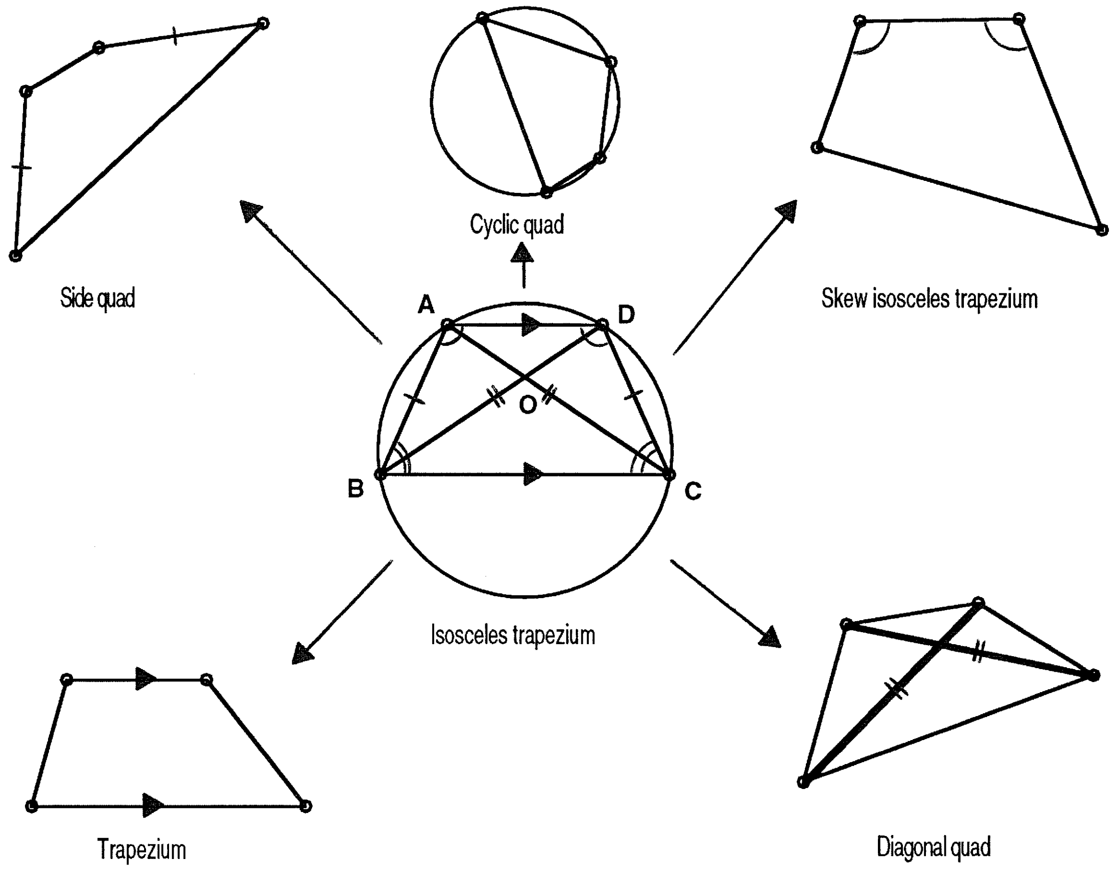


Figure 4.7

The dual of the skew isosceles trapezium is the skew kite. Although there does not seem to be interesting duals between their properties, their duality becomes appparent when we again consider some of their intersections with other quadrilaterals. For example, it is easy to see that a skew isosceles quad that is cyclic is an isosceles trapezium. But a skew kite that is circumscribed around a circle is a kite. (The proofs are straightforward and left to the reader). Also explore other intersections and investigate further dualities on your own.

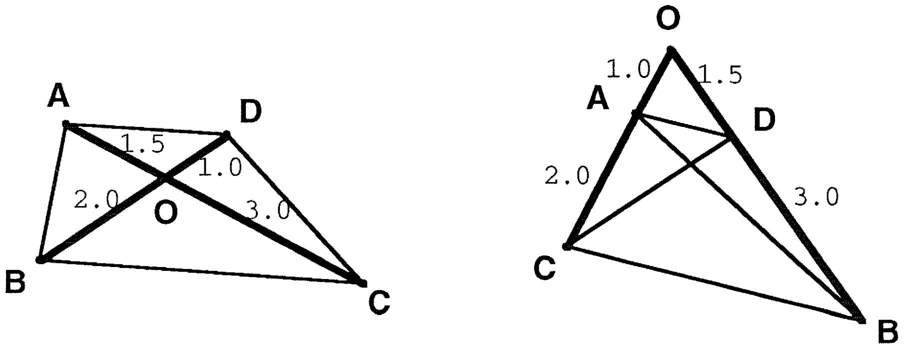


Figure 4.8

Another possible way of generalizing an isosceles trapezium is to consider the ratio into which the diagonals divide each other, namely, AO/OC = DO/OB. Suppose we generalize this property to define a new concept, a *ratio quad*, as any quadrilateral with diagonals dividing each other into the ratio AO/OC = DO/OB. However, if as shown in Figure 4.8 we construct such a quadrilateral, it always turns out to be a trapezium with AD//BC. (Why can't we get a concave ratio quad?).

The proof relies on the similarity of triangles AOD and COB and is left to the reader to complete.

(4) A circum quad with perpendicular diagonals is a kite. Empirical investigation with *Cabri* or *Sketchpad* confirms that the conjecture is indeed true.

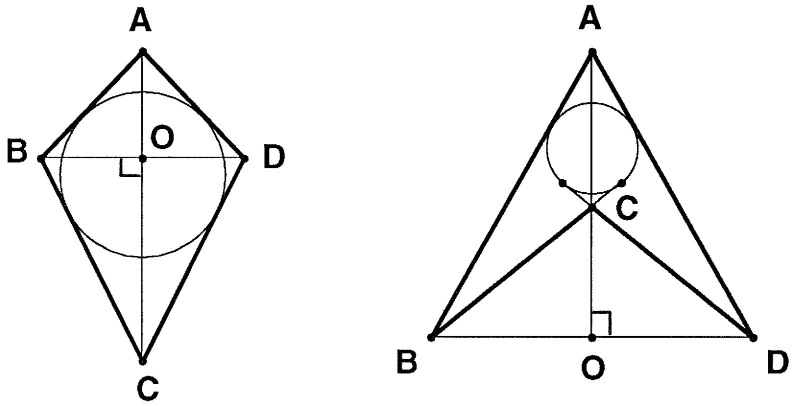


Figure 4.9

Consider Figure 4.9. From Pythagoras we have:

$(1) AO^2 + OB^2 = AB^2$

$(2) AO^2 + OD^2 = AD^2$

$(3) BO^2 + OC^2 = BC^2$

$(4) DO^2 + OC^2 = CD^2$

Subtract (3) from (1) and (4) from (2) to obtain:

$(5) AO^2 - OC^2 = AB^2 - BC^2$

$(6) AO^2 - OC^2 = AD^2 - CD^2$

and therefore we have:

$(7) AB^2 + CD^2 = BC^2 + AD^2$.

(Note that this is a general property of a perpendicular quad, namely, that the sum of the squares of two opposite sides is equal to the sum of the squares of the other two sides. What about the converse, is it also true? Investigate.).

But for a circum quad $AB + CD = BC + AD$, and squaring both sides we obtain:

$(8) AB^2 + 2AB \cdot CD + CD^2 = BC^2 + 2BC \cdot AD + AD^2$.

Subtract (7) from (8) and simplify to obtain:

$(9) AB \cdot CD = BC \cdot AD$.

Now substitute $AB=BC+AD-CD$ in (9) and simplify:

$$CD^2 - CD(AD+BC) + BC \cdot AD = 0$$
$$\therefore (CD-BC)(CD-BC) = 0$$
$$\therefore CD = BC \text{ or } AD.$$

By substitution into (9) we now see that if $CD=BC$, then $AB=AD$, and if $CD=AD$, then $AB=BC$. In either case ABCD is a kite, and therefore completes the proof.

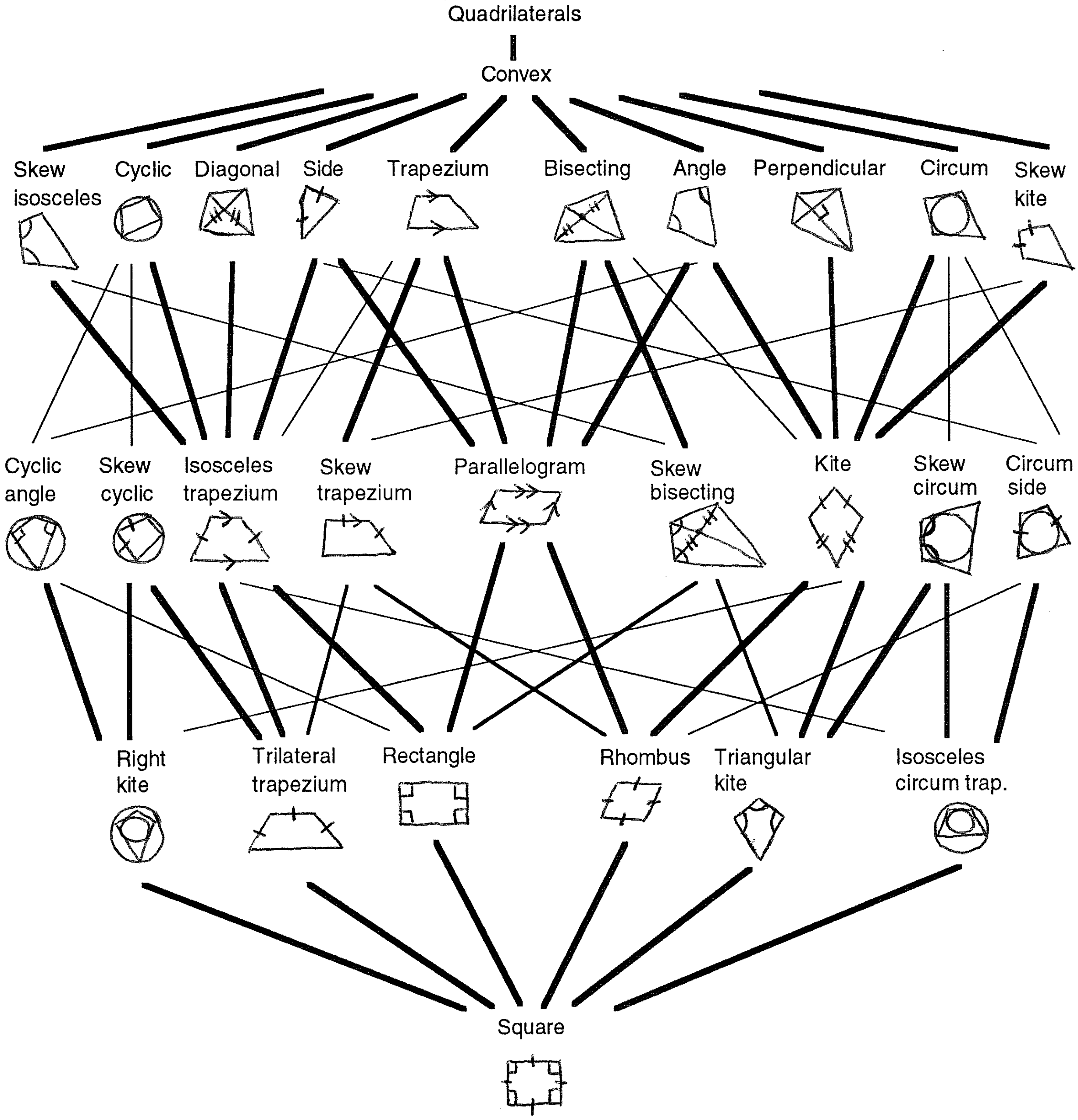


Figure 4.10

(5) A classification scheme for convex quadrilaterals is shown in Figure 4.10, separately from the one for crossed and concave quadrilaterals shown in Figure 4.11. Rough icons have been drawn to enable easier identification. Both figures have been drawn

(more or less) symmetrically around a vertical line in the middle so that the dual of a particular figure can easily be found by reflection in the line of symmetry.

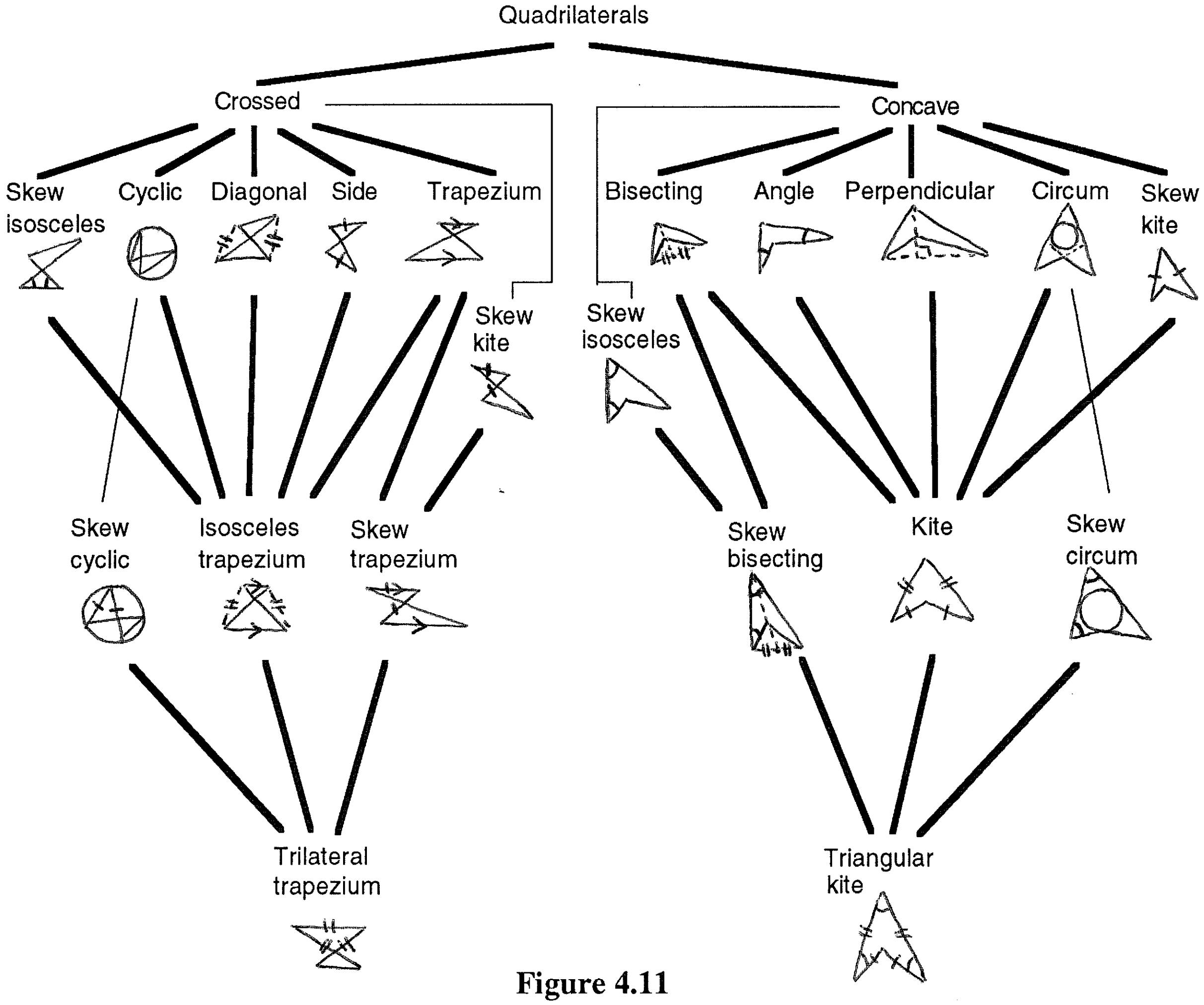


Figure 4.11

Note that in Figure 4.10 the hierarchical class inclusion of the skew cyclic quads under the skew kites, as well as the skew circum quads under the skew isosceles quads were not indicated to avoid further cluttering up the already complex figure. Similarly, due to space limitations concave diagonal quads and their dual, namely, crossed perpendicular quads were not included in Figure 4.11. Also note that although it is possible to construct a concave side quad, it is not possible to obtain a dual, namely, a crossed angle quad since a crossed quadrilateral cannot have opposite angles equal. In addition, it is easy to define several more different quadrilaterals and to consider their intersections with those above, as well as possible duals. Some examples are a *bi-diagonal* quad (see Chapter 8 (cont.), Fig. 8.1) and an *angle-bisecting* quad (a quadrilateral with at least one of its angles bisected by a diagonal). Also note that the *right* quad defined in Figure 78 is the same as a cyclic angle quad.

6. (a) Consider Figure 4.12. Since triangles APO and ASO, and CQO and CRO are congruent $(90°, s, s)$, we have $\angle POA = \angle SOA = x$ and $\angle QOC = \angle ROC = y$. Then $\angle POQ = 180° - x - y = \angle SOR$. Therefore isosceles triangles POQ and SOR are

congruent $(s,\angle,s)$. Therefore $\angle BPQ = \angle BQP = 90° - x/2 - y/2 = \angle DSR = \angle DRS$ which implies isosceles triangles PBQ and SDR are congruent $(\angle,\angle,s)$. Therefore BQ+QC= DR+RC which implies BC=DC. Similarly, AB=AD and therefore ABCD is a kite with AC as an axis of symmetry. Similarly it can be shown that BD is an axis of symmetry, and therefore ABCD must be a rhombus.

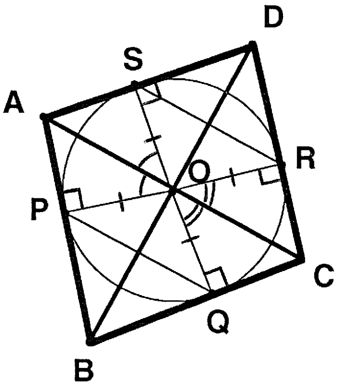


Figure 4.12

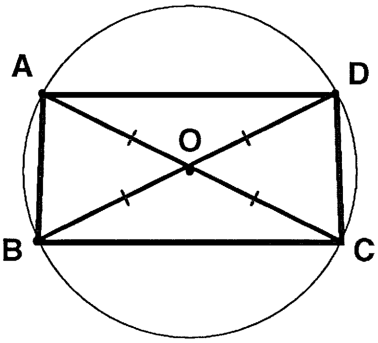


Figure 4.13

(b) If the diagonals of a cyclic quad intersect at its circumcentre, then it is a rectangle. Consider Figure 4.13. Isosceles triangles AOD and BOC, as well as AOB and DOC, are congruent $(s,\angle,s)$. Therefore opposite sides are equal and ABCD is a parallelogram, but a parallelogram that is cyclic is a rectangle.

This particular proof is another good example of utilizing a hierarchical classification as discussed in Chapter 2.

7. (a) The points A, O and P are collinear (lie in a straight line). The result can be formulated as follows: If O is the incentre of triangle ABC, then the circumcentre of triangle BOC, say P, lies on the angle bisector of angle A, namely line AO.

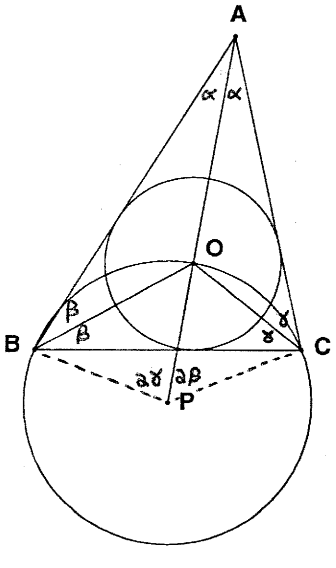


Figure 4.14

Proof

Consider Figure 4.14. Since P is the circumcentre of triangle BOC, we have PB=PO=PC. we shall prove that POA is a straight line, by showing that $\angle POB = \angle OBA + \angle OAB$.

Let $\angle A = 2\alpha$, $\angle B = 2\beta$ and $\angle C = 2\gamma$. Then $\alpha + \beta + \gamma = 90°$. Using the "angle at the centre" theorem we deduce that $\angle OPC = 2\beta$ and $\angle OPB = 2\gamma$. So $\angle BPC = 2(\beta + \gamma)$ and since triangle BPC is isosceles, we have:

$\angle CBP = \angle BCP = 180° - \angle BPC = 90° - \beta - \gamma = \alpha$.

So $\angle POB = \angle PBO = \alpha + \beta = \angle OBA + \angle OAB$, as required.

(b) If O is the circumcentre of triangle ABC, then the incentre of triangle BOC, say P, lies on the perpendicular bisector of BC (the side opposite angle A), namely line EO, where E is the midpoint of BC.

Is it true? Investigate. (See *Solutions 4 (continued)*).

8. (a) The orthocentre, centroid and circumcentre of any triangle are collinear. The line on which they lie is called the *Euler line*, after the famous Leonhard Euler (1707-1783).

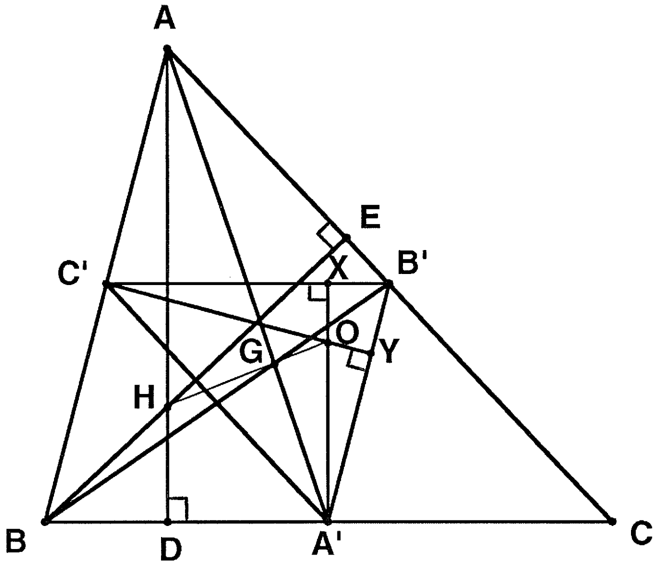


Figure 4.15

Proof

Consider Figure 4.15. The triangle formed by joining the midpoints of the sides of a given triangle is called the *medial* triangle. In the given figure triangle A'B'C' is the medial triangle of triangle ABC. Draw the medians AA' and BB' of ΔABC to meet at G, as well as its altitudes AD and BE to meet at H. Also draw the altitudes A'X and C'Y of ΔA'B'C' to meet at O.

Now we have ΔA'B'C' similar to ΔABC since their corresponding sides are parallel. Next, C'B' = ½BC, so the ratio between any two corresponding line segments (not merely corresponding sides) will be 1:2.

Next, note that AC'A'B' is a parallelogram so that AA' bisects B'C'. Therefore, the medians of ΔA'B'C' lie along the medians of ΔABC, which means that both triangles have the same centroid, G.

Furthermore, the altitudes of ΔA'B'C' we have drawn are the perpendicular bisectors of the sides AB and BC of ΔABC. Therefore O, the *orthocentre* of ΔA'B'C', is at the same time the *circumcentre* of ΔABC.

Since H is the orthocentre of ΔABC while O is the orthocentre of the similar triangle A'B'C', AH = 2OA'. Furthermore, since G is the centroid of triangle ABC, we have AG = 2GA'. Finally, since AD and OA' are both perpendicular to the side BC, they are parallel. Hence, $\angle HAG$ is equal to alternate $\angle OA'G$. Therefore, triangles HAG and OA'G are similar (two corresponding sides in same ratio and enclosed angle is equal). This shows that $\angle AGH = \angle A'GO$ which implies that HGO is a straight line, i.e. O, G and H are collinear.

Looking back

Note further from the similarity of triangles HAG and OA'G that HG=2GO so that the centroid divides the distance from the orthocentre to the circumcentre in the ratio 2:1.

(b) A possible dual is: the orthocentre H, centroid G and *incentre* O of any triangle are collinear. Unfortunately this dual turns out to be false as shown in Figure 4.16.

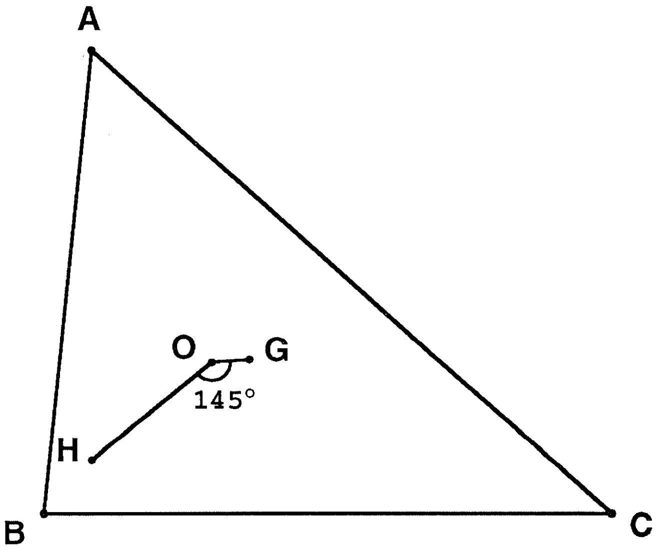


Figure 4.16

However, it is possible to conceptualize another dual as follows:

- *circumcentre* has as dual *incentre*
- *orthocentre* is selfdual since the concept *altitude* is selfdual (shortest distance from *vertex* to opposite *side* is dual to the shortest distance from *side* to opposite *vertex*)
- *centroid* seems to have as dual also the *incentre* since *medians* and *angle bisectors* appear to be dual (a median is the bisecting *point* of a *side* connected to the opposite *vertex* and a bisector is the bisecting *line* of an *angle* connected to the opposite *side*, if extended).

However this dual of the Euler line is trivial, namely, that the incentre and orthocentre of any triangle are collinear, but that is obviously true for any two points.

9. (a) Consider Figure 58 in the text. Diagonal PR=½AB in ΔABD and diagonal QS=½AD in ΔACD. But it is given that AB=AD; therefore PR=QS. Is the result also true for crossed skew kites?

(b) A possible dual might be: In any skew isosceles trapezium with $\angle A = \angle D$, P and S the midpoints of sides AD and DC and Q and R are the midpoints of the diagonals AC and BD, then PQRS is a perpendicular quad.

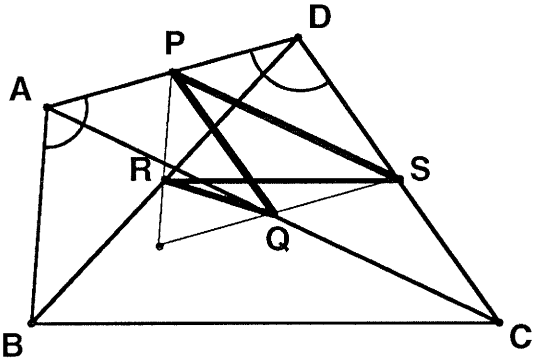


Figure 4.17

As shown in Figure 4.17, this dual is unfortunately not true. However, it is easy to see that if it is given that $\angle A = \angle D = 90°$, then PR⊥QS so that PQRS would be perpendicular quad.

10. (a) (i) This result is almost an immediate consequence of the Fermat generalization given in Figure 3.12 in *Solutions 3 (continued)*, no. 3, and is left to the reader to complete.

(ii) This result also follows from the same generalization given in (i).

(b) The respective duals are:

(i) If similar isosceles triangles DBA, ECB and FAC are erected on the sides of any ΔABC, then the linesegments (GC, HA and IB) connecting their *incentres* with the opposite vertices of the base triangle are concurrent (see Figure 4.18). This result also follows from the Fermat generalization given in (a)(i) above.

(ii) If similar triangles ABD, EBC and AFC are erected on the sides of any ΔABC, then the linesegments connecting the *circumcentres* with the opposite vertices of the base triangle are concurrent. Unfortunately this dual is not true as shown by the counter-example in Figure 4.19.

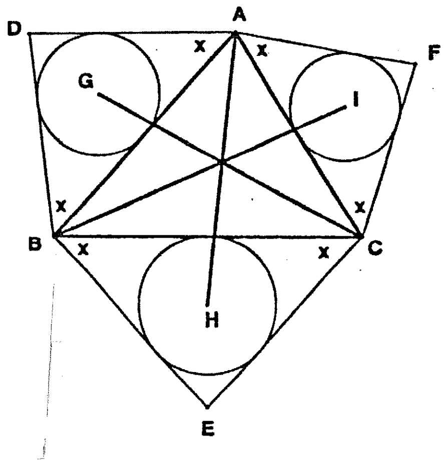


Figure 4.18

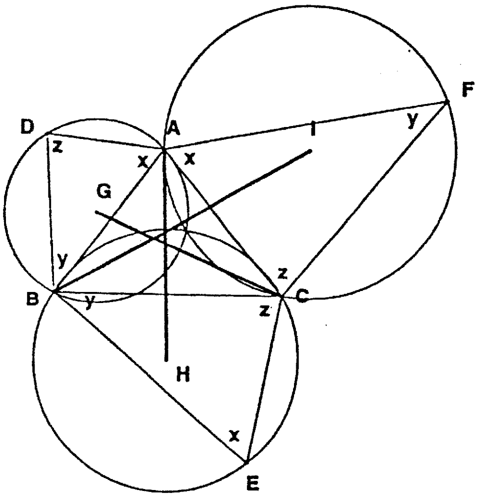


Figure 4.19

11. (a) The diagonals of the circum quad ABCD and its associated cyclic quad PQRS are concurrent.

This result is a special case of Brianchon's theorem shown in Figure 46b. By permitting the sides of the circumscribed hexagon to coalesce and labelling them carefully, we can obtain interesting special cases regarding circumscribed pentagons, quadrilaterals and triangles. In such cases the common vertex of two coincident sides becomes their tangential point (point of contact) with the circle.

Consider, for instance, the circumscribed pentagon ABCDE shown in Figure 4.20a. By regarding it as a degenerate hexagon with a straight angle at F, we can apply Brianchon's theorem. Similarly, we can apply Brianchon's theorem to the special cases shown in Figures 4.20b-d. The above formulated result is given in Figure 4.20c and is the consequence of the formation of two degenerate circumscribed hexagons, namely, ABCDEF and APBDQE.

(b) To obtain a dual to the above result we need to consider limiting cases of the theorem of Pascal mentioned in Chapter 4. For example, assume that the vertex F of a cyclic hexagon ABCDEF moves on the circle and approaches the point E. Then side EF tends to the tangent to the circle at E, and in the limit we obtain the following result:

"The point of intersection of side BC of a cyclic pentagon ABCDE with the tangent at E is collinear with the intersection of sides AB and DE, and CD and AE". (See Figure 4.21a).

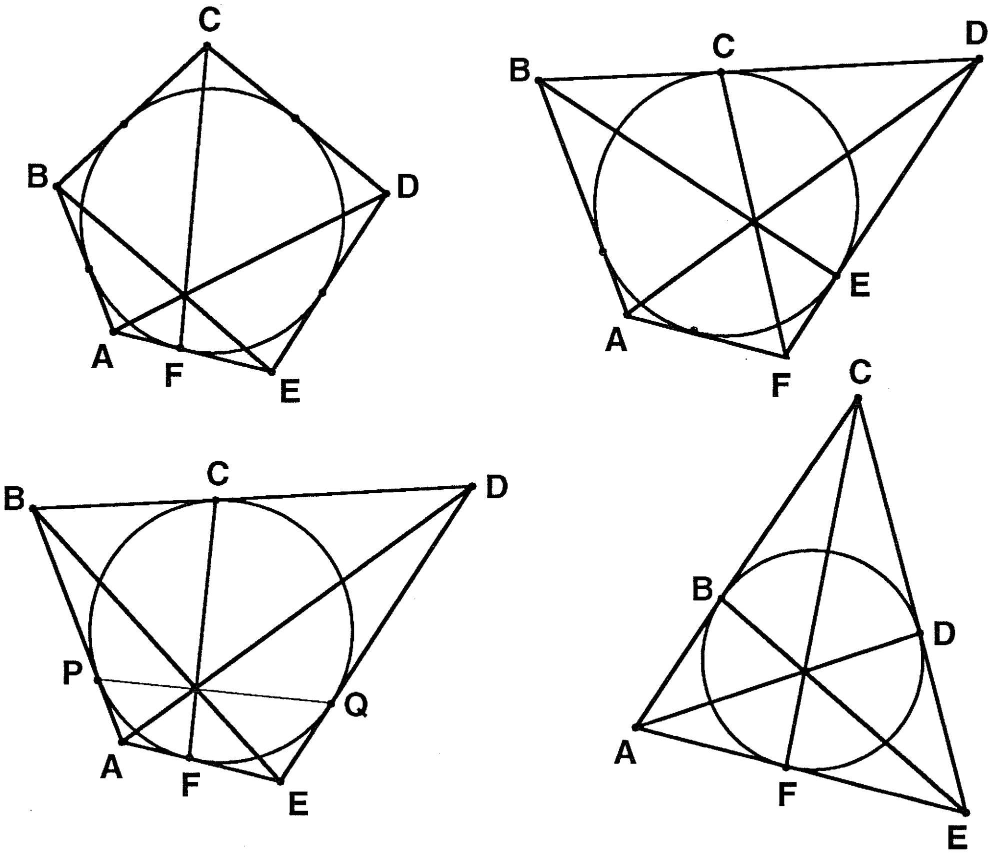


Figure 4.20

Similarly, assuming that in the cyclic hexagon ABCDEF the vertex F coincides with E and the vertex D with C, we obtain the result:

"The point of intersection of sides AB and CE of a cyclic quadrilateral ABCE is collinear with the point where BC meets the tangent to the circle at E, and the point where AE meets the tangent to the circle at C". (See Figure 4.21b).

If, in the hexagon, we assume coincidence of the vertices F and E, and C and B, then we readily see that the point of intersection of the tangents to a circle at the vertices E and B of a cyclic quadrilateral ABDE is collinear with the points of intersection of the opposite sides; it should also be clear that the point of intersection of the tangents to the circle at the points A and D also lie on this line (see Figure 4.21c).

Finally, the assumption that the vertices A and B, C and D, E and F of the cyclic hexagon coincide implies that the points of intersection of the sides of a triangle ACE with the lines tangent to the circumscribed circle at the opposite vertices are collinear (see Figure 4.21d).

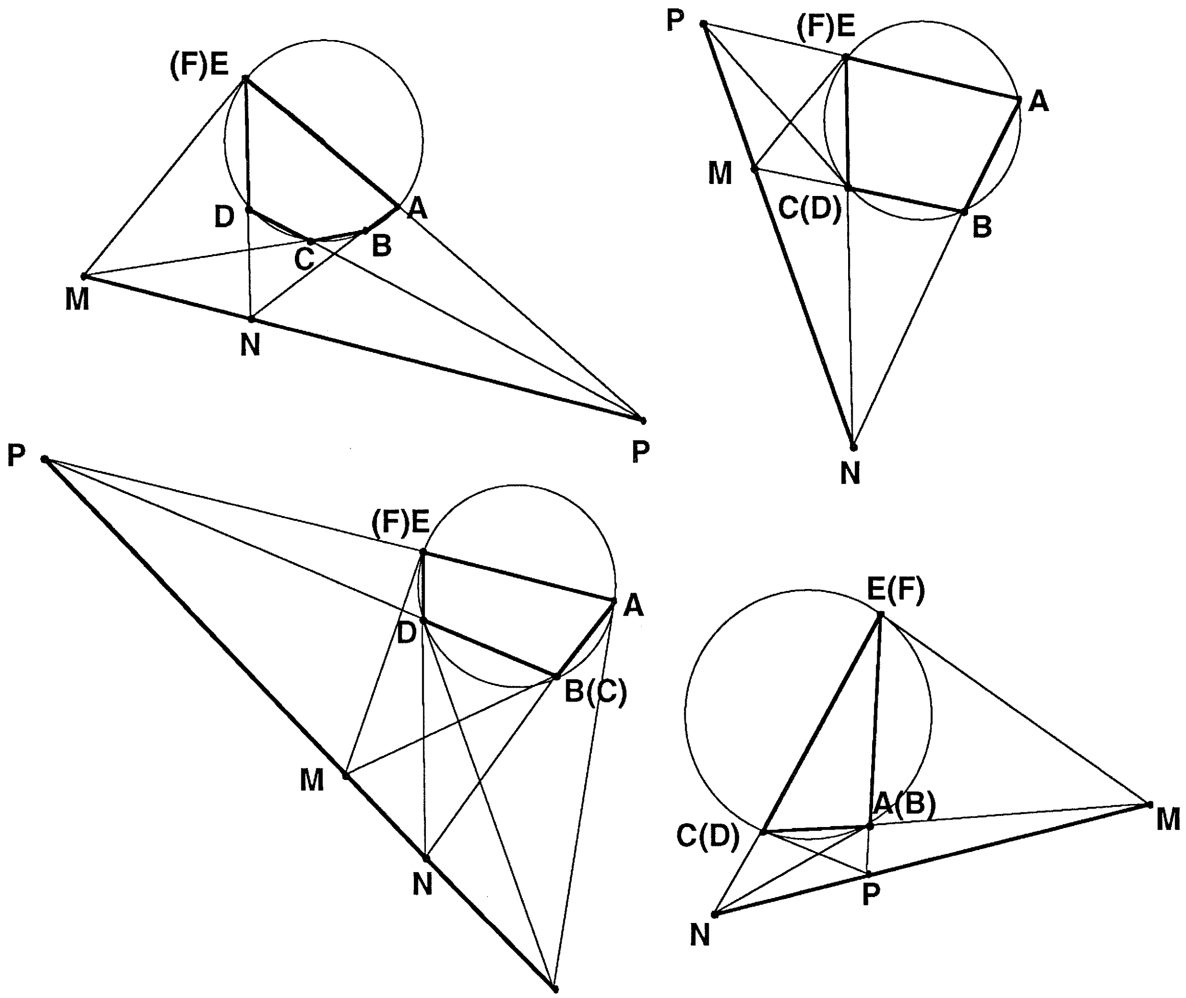


Figure 4.21

12. The points of intersection of the adjacent *perpendicular trisectors* of the sides of any triangle are the vertices of an equilateral triangle.

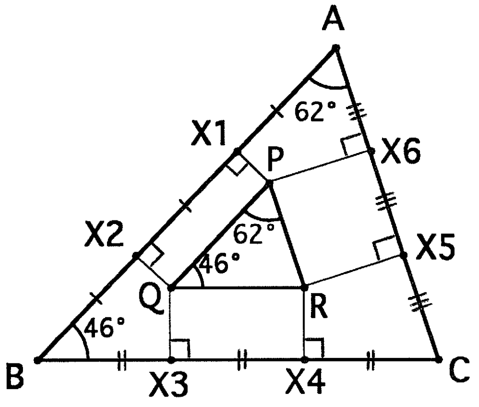


Figure 4.22

Although the dual is unfortunately not true as shown in Figure 4.22, it appears that ΔPQR is similar to ΔABC. Is it always true or not? Investigate. (See *Solutions 4 (continued)*).

13. (a) The three circles intersect in one point. This interesting result is known as *Miquel's theorem* after Miquel who explicitly stated and proved it in 1838, although it was probably known earlier.

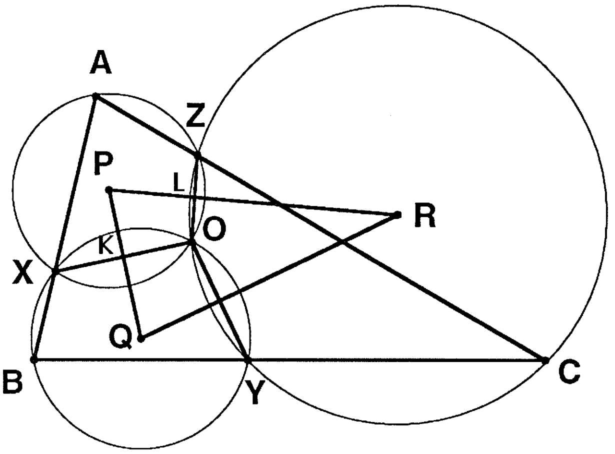


Figure 4.23

Consider Figure 4.23. Let O be the point of intersection of the two circles AYZ and BYX. Then we have $\angle XOZ = 180° - \angle A$ and $\angle XOY = 180° - \angle B$. Therefore $\angle YOZ = \angle A + \angle B = 180° - \angle C$ which shows that the points Z, O, Y and C are concyclic. (What happens if O falls outside ΔABC? Is the result and proof still valid? Investigate).

(b) The ΔPQR is similar to ΔABC. Since PQ and PR are respectively perpendicular on the common chords XO and ZO, it follows that PKOL is a cyclic quadrilateral with $\angle KPL = 180° - \angle XOZ = \angle A$. In the same way, it can be shown that $\angle Q = \angle B$ and $\angle R = \angle C$.

(c) If *angle dividers* for each *angle* of a triangle are drawn to meet in a point and *incircles* are drawn to each side and the adjacent angle dividers, then the *incentres* of these circles form a triangle similar to the original triangle.

Unfortunately this conjecture is not true as shown in Figure 4.24. The reader may also wish to verify that even if we construct circumcircles to triangles AOB, BOC and AOC, then the three circumcentres do not form a triangle similar to the original one.

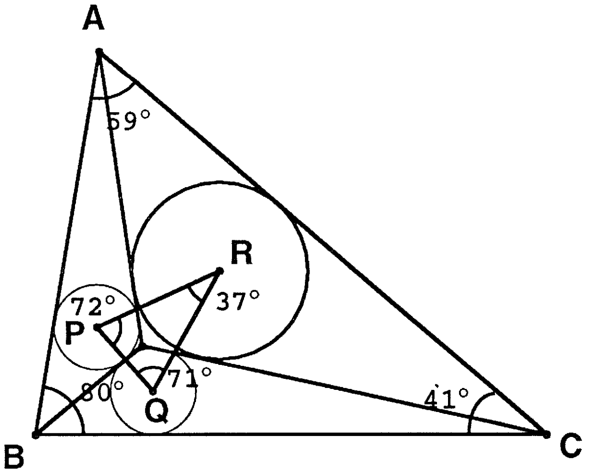


Figure 4.24

14. We can conjecture the following duals to Napoleon's theorem and its generalizations:

(i) If equilateral triangles are erected on the sides of any triangle, their *incentres* form an equilateral triangle.

(ii) If similar triangles ADB, CBE and FAC (or DBA, CBE and CFA) are erected on the sides of any ΔABC, their *incentres* form a triangle similar to the three triangles.

(iii) If triangles ADB, CBE and FAC are erected on the sides of any ΔABC so that $\angle D+\angle E+\angle F=180^\circ$, their *incentres* form a triangle with $\angle G=\angle D$, $\angle H=\angle E$ and $\angle I=\angle F$.

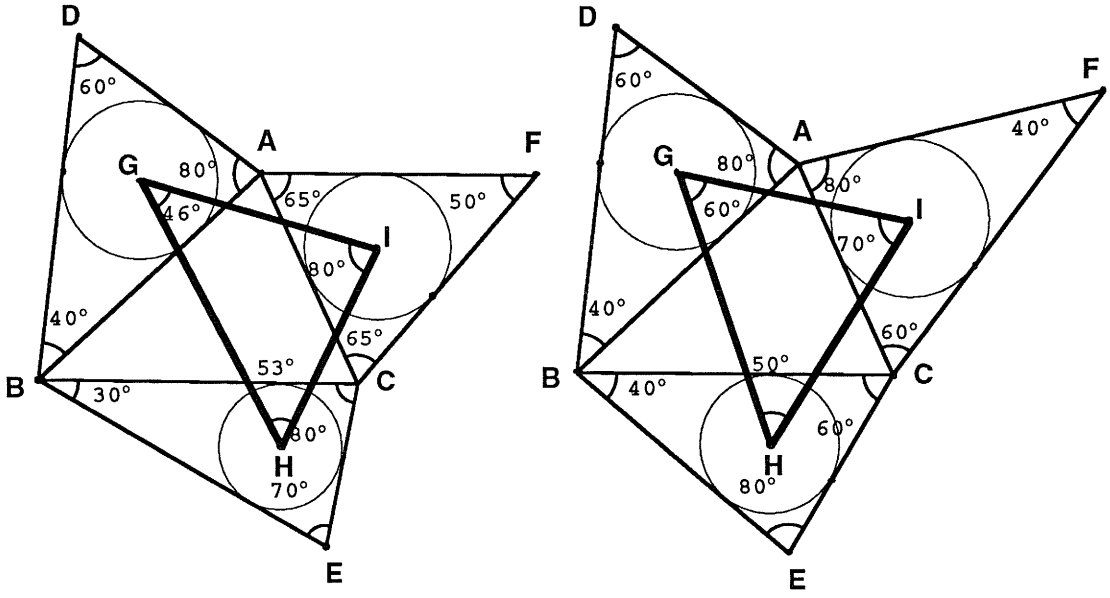


Figure 4.25

As the circumcentre and incentre of any equilateral triangle coincide, the first conjecture is simply a corollary of Napoleon's theorem which involves the circumcentres of the equilateral triangles. Napoleon's theorem is therefore *self-dual* with respect to the concepts circumcentre and incentre.

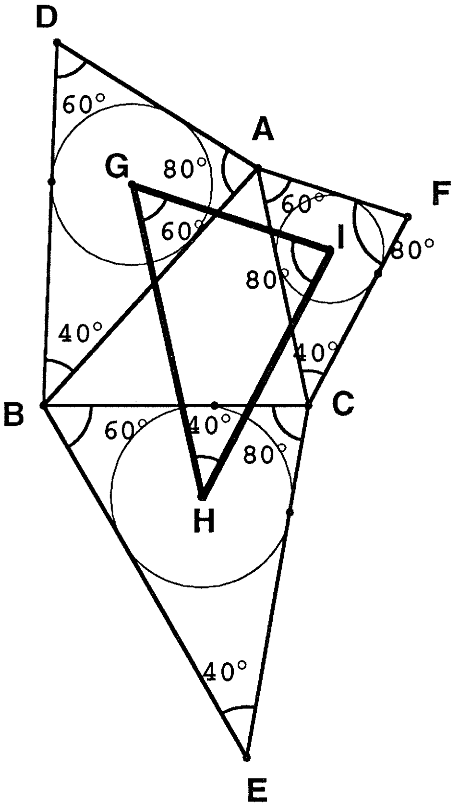


Figure 4.26

As shown in Figure 4.25a, the third conjecture is unfortunately false. Furthermore, if we construct similar triangles DBA, CBE and CFA we do not obtain a triangle GHI similar to the three triangles (see Figure 4.25b). However, it appears that in this case the incentres form a triangle with $\angle G = \frac{1}{2}(\angle DAB + \angle DBA)$, $\angle H = \frac{1}{2}(\angle EBC + \angle ECB)$ and $\angle I = \frac{1}{2}(\angle FCA + \angle FAC)$. Is this always true or not? Investigate. (See *Solutions 4 (continued)*).

Furthermore, if we construct similar triangles ADB, CBE and FAC as shown in Figures 4.26, then triangle GHI is similar to the three triangles. Is this always true or not? Investigate. (See *Solutions 4 (continued)*).

15. (a) The lines DC, EA and FB are concurrent. This result is also a consequence of the Fermat generalization given in Figure 3.10.

    (b) What happens if we have similar rectangles or rhombi on the sides? (See *Solutions 4 (continued)*).

16. We can define as dual to the kite $2n$-gon the concept "*isosceles $2n$-gon*" as a $2n$-gon with a line of symmetry through one pair of opposite sides as shown in Figure 4.27.

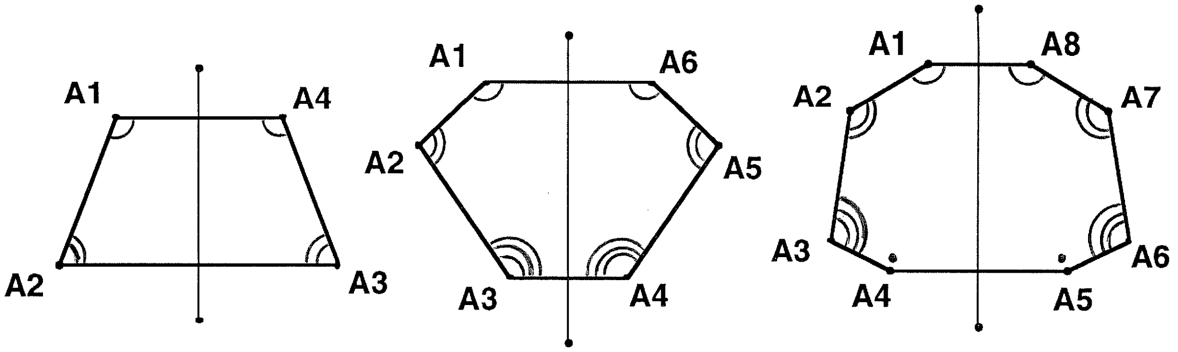


Figure 4.27

We then have the following dual to the equality of the two sums of alternate sides of a kite-$2n$-gon: "In an isosceles $2n$-gon the two sums of alternate angles are equal."

Proof

Due to symmetry we have $\angle A_1 = \angle A_{2n}, \angle A_2 = \angle A_{2n-1}, \angle A_3 = \angle A_{2n-2}$, etc. Therefore: $\angle A_1 + \angle A_3 + \ldots + \angle A_{2n-1} = \angle A_2 + \ldots + \angle A_{2n-2} + \angle A_{2n}$.

17. (a) The two sets of alternate angles of a convex cyclic hexagon are equal to 2π (2π=360°). This result was discovered by a student named Turnbull during a classroom activity (see Mackinnnon, 1990).

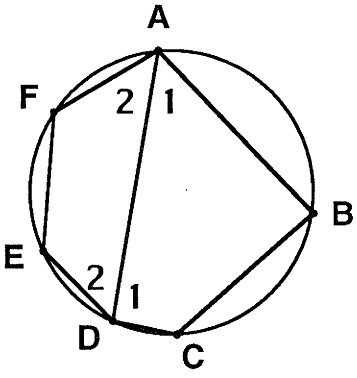


Figure 4.28

Proof

Consider Figure 4.28. Connect A with D. Then we have $\angle A_1 + \angle C = \angle B + \angle D_1 = \pi$ and $\angle A_2 + \angle E = \angle D_2 + \angle F = \pi$. By addition we obtain the desired result, namely: $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F = 2\pi$.

(b) This result can be generalized as follows to any convex cyclic $2n$- gon ($n > 1$) since it can be divided into n convex cyclic quadrilaterals (and can therefore be proved by mathematical induction in the same way as for the convex cyclic hexagon):

"If $A_1A_2...A_{2n}(n > 1)$ is any convex cyclic polygon P, then the two sums of alternate interior angles of P are each equal to $(n-1)\pi$."

(c) Consider a convex circum hexagon as shown in Figure 4.29. It is not difficult to see that since the tangents from a point to a circle are equal, we would obtain: AB + CD + EF = BC + DE + FA.

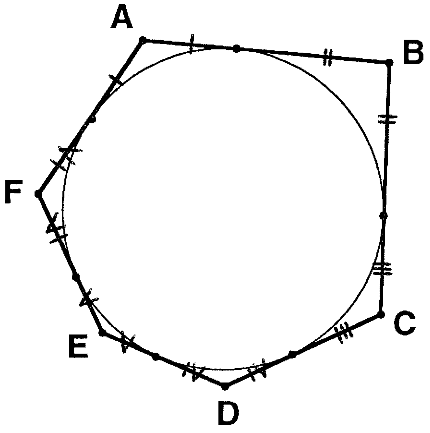


Figure 4.29

This result is obviously dual to the preceding one, and can be formulated as follows: "If $A_1A_2...A_{2n}(n > 1)$ is any convex circumscribed polygon P, then the two sums of alternate sides of P are equal."

Can you generalize these two dual results further? (Try investigating other types of $2n$-gons - see *Solutions 4 (continued)*).

(d) Unfortunately the converses are only valid for cyclic and circumscribed quadrilaterals (n = 2). The first figure in Figure 4.30, for example, shows a convex hexagon for which the two sums of alternate sides are equal, but a circle cannot be inscribed in it (the angle bisectors are not concurrent). Similarly, the second figure in Figure 4.30 shows a convex hexagon with the two sums of alternate angles equal, but a circle cannot be circumscribed around it (the perpendicular bisectors of the sides are not concurrent).

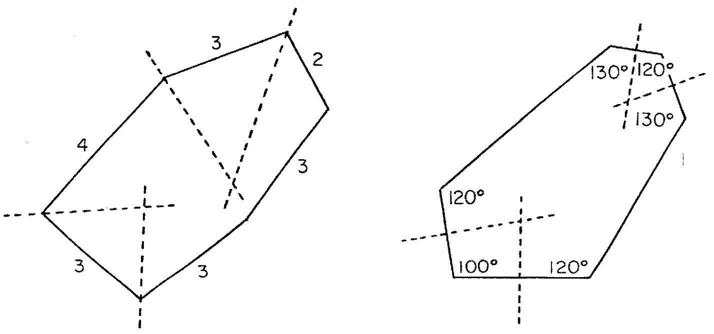


Figure 4.30

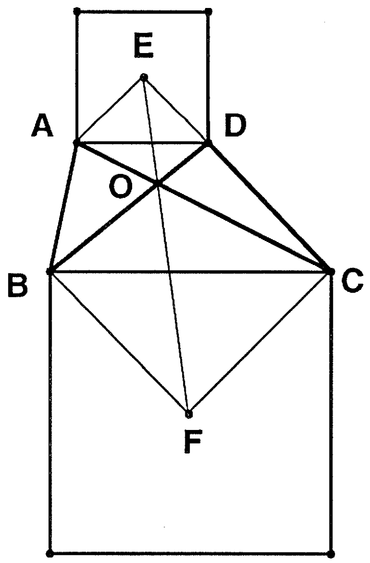


Figure 4.31

18. (a) The points E, O and F are collinear. Consider Figure 4.31. Draw the line segments as shown. Firstly, ΔAOD is similar to ΔCOB $(\angle, \angle)$ which implies:

$$\frac{AO}{OC} = \frac{AD}{CB} \ldots (1).$$

Secondly, from the construction we have isosceles ΔAED similar to isosceles ΔCFB and therefore:

$$\frac{AD}{BC} = \frac{AE}{CF} \ldots (2).$$

From (1) and (2) we therefore have AO/OC = AE/CF. But $\angle EAO = \angle FCO$ and therefore triangles AOE and COF are similar $(s, \angle, s)$. Therefore $\angle AOE = \angle COF$, and since AC is a straight line, this implies that EOF is a straight line as well.

Is the result also true for crossed trapezia? Investigate. What happens if the squares are constructed inward? Is the result still valid?

(b) What happens if we constructed similar rectangles or rhombi instead? Investigate. (See *Solutions 4 (continued)*).

19. Yes, they are true as shown in Figure 4.32. The proofs are left to the reader.

20. (a) EFGH is a kite, since the figure as a whole has a line of symmetry through H and F, and therefore a reflection around HF will map E onto G. Is the result also true for a crossed isosceles trapezium? What happens if the squares are constructed inward?

(b) If squares are constructed on the sides of any *kite*, then the centres of the squares form an *isosceles trapezium* (see Figure 4.33).

As before, this result follows directly from the line of symmetry through A and B of the whole figure. Is the result also true for a concave kite? What happens if the squares are constructed inward?

(c) As shown in Figure 4.34, we obtain a quadrilateral with equal and perpendicular diagonals, i.e. a *diagonal perpendicular* quad. (Note therefore that the kite in Figure 66 and the isosceles trapezium in Figure 4.33 are also respectively diagonal and perpendicular quads). This result can easily be verified with the property checker of *Cabri-geometre* to be true in general.

Can you explain *why* this result is true? What happens if the squares are constructed inward? What if the base quadrilateral is concave or crossed?

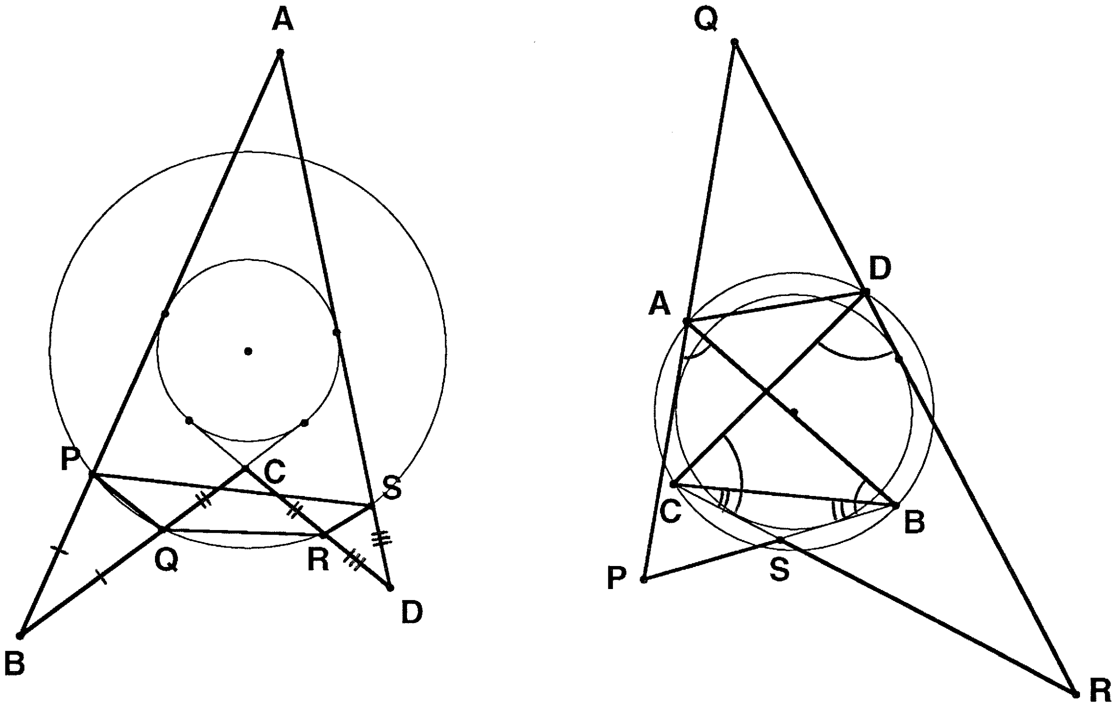


Figure 4.32

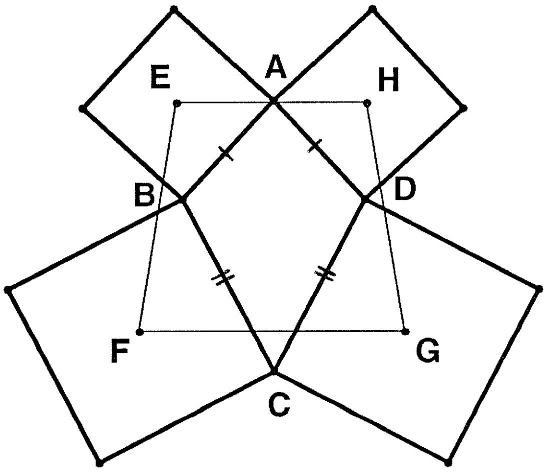


Figure 4.33

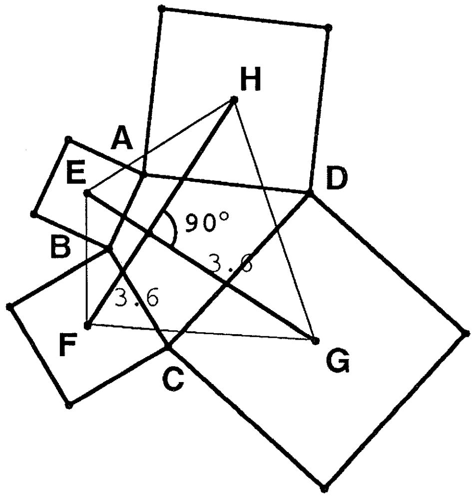


Figure 4.34

21. (a) PQRS is also an isosceles trapezium. The result follows directly from symmetry. For example, if P and S are reflected around the axis of symmetry of ABCD they would respectively map onto Q and R.

(b) The respective centroids P, Q, R and S of triangles ABC, BCD, CDA and DAB of a kite ABCD also form a kite (see Figure 4.35). This result also follows directly from symmetry. In the preceding case, as well as this one, compare the length and direction of the sides of PQRS with those of ABCD - what do you notice?

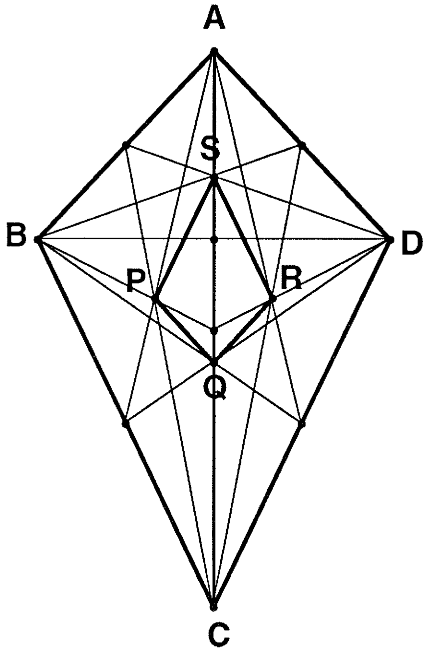


Figure 4.35

Are these results also true for crossed isosceles trapezia and concave kites? What happens if ABCD is a rectangle or a rhombus? What happens if ABCD is any quadrilateral? What happens if instead of centroids we connect say, the incentres or circumcentres of triangles ABC, BCD, CDA and DAB?

Investigate the above questions (see *Solutions 4 (continued)*).

22. (a) Yes, it is sufficient - the deductive explanation is left to the reader.
(b) The dual of this result is also true, namely, if three of the perpendicular bisectors of the sides of a quadrilateral are concurrent, then it is cyclic. As above, the deductive explanation is left to the reader.

23. (a) Consider Figure 69. $\angle HEF = 180° - \frac{1}{2}\angle A - \frac{1}{2}\angle B$ and $\angle HGF = 180° - \frac{1}{2}\angle C - \frac{1}{2}\angle D$. Therefore $\angle HEF + \angle HGF = 360° - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$, but $\angle A + \angle B + \angle C + \angle D = 360°$. Therefore $\angle HEF + \angle HGF = 180°$, from which follows that EFGH is a cyclic quadrilateral.

Is the result still valid for a crossed quadrilateral? What happens if we construct the external angle bisectors of a quadrilateral as shown in Figure 4.36? (See *Solutions 4 (continued)*).

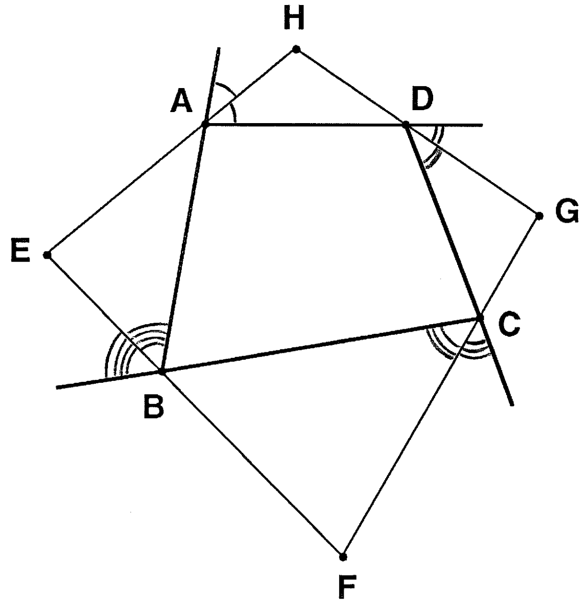


Figure 4.36

(Note that the angle bisectors of a circum quad are concurrent, in which case the formed cyclic quadrilateral collapses into a single point).

(b) One could conjecture the following dual: "The perpendicular bisectors of the sides of any quadrilateral form a circum quad."

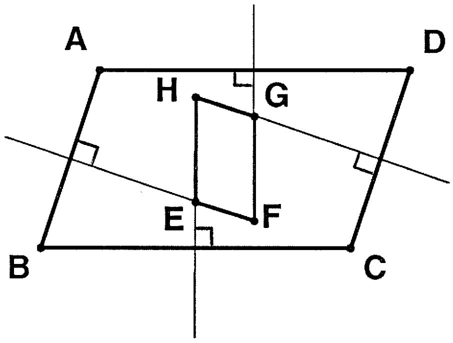


Figure 4.37

This is unfortunately not true for a parallelogram as shown in Figure 4.37 where the formed quadrilateral EFGH is also a parallelogram. Furthermore, the perpendicular bisectors of the sides of a cyclic quadrilateral are concurrent, so no quadrilateral is formed. However, it is easy to see that it would be true for a rhombus and a kite, which leads us naturally to the following conjecture: "The perpendicular bisectors of the sides of any circum quad form another circum quad."

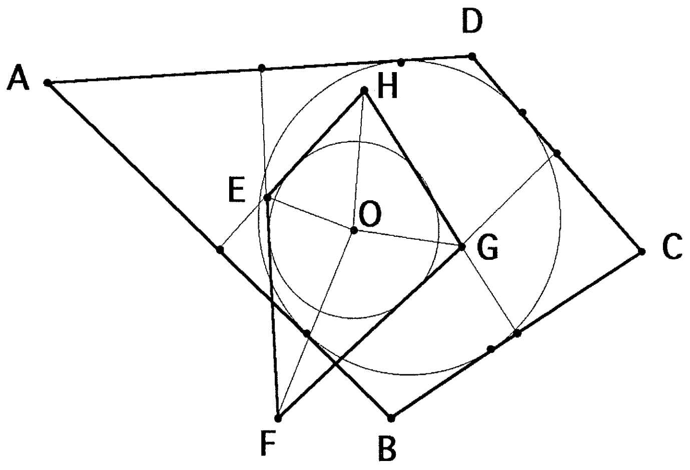


Figure 4.38

Investigation of a convex circum quad ABCD with the property checker of *Cabri-geometre* as shown in Figure 4.38, confirms that the angle bisectors of EFGH are always concurrent and therefore that the conjecture is true in general. Can you provide a satisfactory *explanation* (proof) of why it is true? (See *Solutions 4 (continued)*).

24. (a) Hint: Place the equilateral triangle with its incentre at the origin of coordinate axes and use coordinate geometry. (See *Solutions 4 (continued)*).

(b) Dual: If Q is any arbitrary point on the circumcircle of an equilateral ΔABC, then $h_{AB}^2 + h_{BC}^2 + h_{AC}^2$ is a constant (where h_i are the respective distances to the three sides). Is it true or not? Investigate. (See *Solutions 4 (continued)*).

25. (a) Repeat the given constructions respectively for circumscribed and cyclic hexagons. What do you find? (See *Solutions 4 (continued)*).

(b) Repeat the given constructions on a triangle, going around twice instead of only once. What do you find? (See *Solutions 4 (continued)*).

26. If triangles DBA, ECB and FAC are constructed outwardly on the sides of any ΔABC so that DA = FA, DB = EB and EC = FC, then the perpendicular from D to AB (side opposite C), the perpendicular from E to BC (side opposite A) and the perpendicular from F to AC (side opposite B) are concurrent. (Note: In the Fermat-Torricelli configuration a pair of equal *angles* are constructed adjacent to each angle and the shortest distances from the top vertices to the opposite *angles* of the base triangle are drawn. In the above formulation, a pair of equal *sides* are constructed adjacent to each angle and the shortest distances from the top vertices to the *sides* opposite the opposite angles of the base triangle are drawn).

Is it true or not? Investigate. (See *Solutions 4 (continued)*).

Solutions 4 (continued)

1. (b) If the incircle of a triangular kite is constructed, then two equal adjacent sides (next to the central equal angle) are bisected by the tangential points E and H as shown in Figure 4.39a. This follows directly from the result that the two pairs of tangents from two points to the same circle are equal, if the angles which the pairs of tangents form at each point are equal (see Figure 4.39b). For example, triangles ABO and XYO are congruent ($\angle, \angle, s$) and therefore AB=XY.

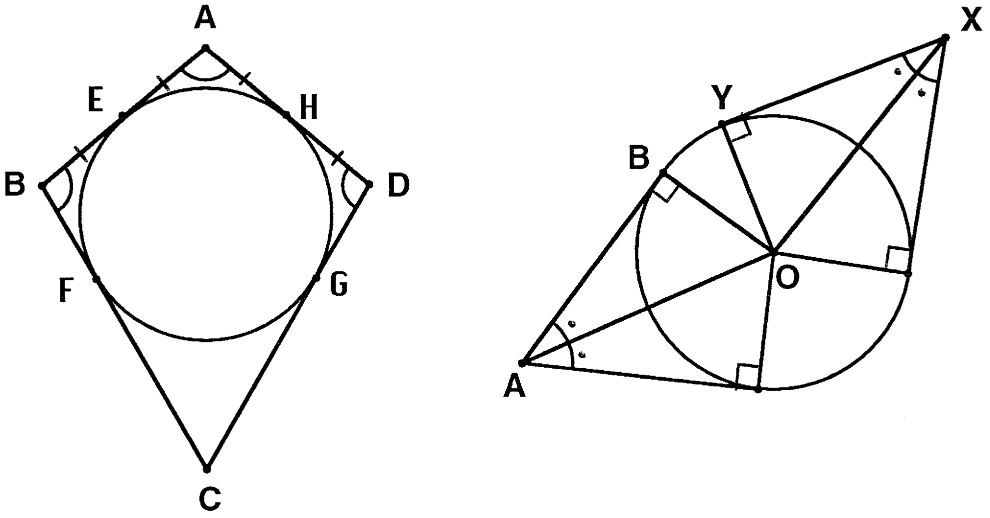


Figure 4.39

The right kite and isosceles circumtrapezium also nicely reflect this duality between angle bisection by diagonal and side bisection by tangential point. For example, as shown in Figure 4.1 the one *diagonal* of a right kite bisects one pair of opposite *angles*, while one pair of opposite *sides* of the isosceles circumtrapezium is bisected by the *tangential points* of the incircle.

(d) As above in 1(b), if the incircle of a skew circum quad is constructed then the side included by the two equal adjacent angles, is bisected by the tangential point of the incircle (see Figure 4.5a). Also note that as shown in Figure 4.40a that this result is also true for concave skew circum quads (tangential point E bisects AB), but that the dual result is not valid for crossed skew cyclic quads as shown in Figure 4.40b.

Formulate converses for both results and investigate whether they are true or not.

2. (a) A convex side quad with equal diagonals is indeed an isosceles trapezium (see Figure 4.41a). From the congruency of triangles ABD and DCB (s,s,s) we have $\angle A = \angle D$. Also $\angle CAD = \angle BAD$ implies $\angle BAC = \angle CDB$ and therefore triangles BAC and CDB are congruent ($s, \angle, s$). Therefore $\angle B = \angle C$, etc.

However, a crossed side quad with equal diagonals is not necessarily an isosceles trapezium as shown in Figure 4.41b where it is easy to verify that ABDC is a parallelogram. On the other hand, if the two *crossing* opposite sides are equal, then ABCD is an isosceles trapezium as shown in Figure 4.41c (the proof is left to the reader).

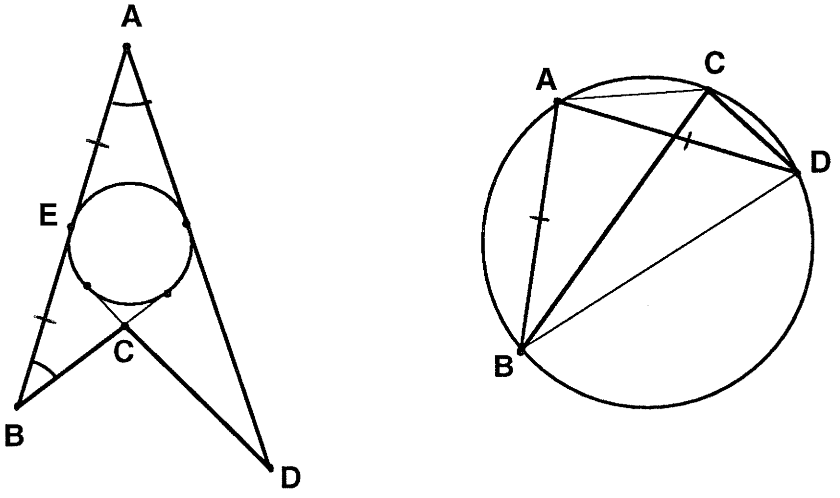


Figure 4.40

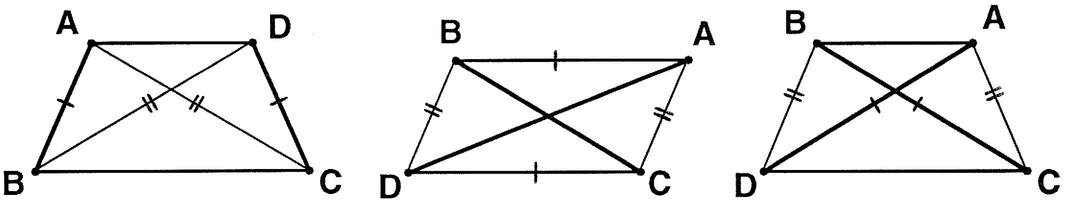


Figure 4.41

(b) True, an angle quad circumscribed around a circle is a kite. Consider Figure 4.42. From the congruency of isosceles triangles BEF and DHG, we have EB=HD and BF=DG which implies AB=AD and CB=CD.

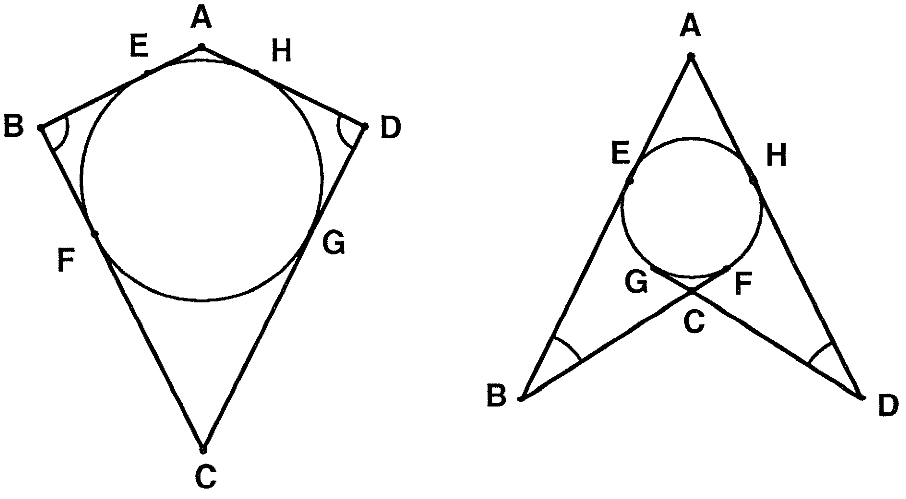


Figure 4.42

(c) True; empirical investigation on *Cabri* or *Sketchpad* shows that a bisecting quad circumscribed around a circle is indeed a kite. Let's try and use the same approach as in *Solutions 4*, no. 4. Consider Figure 4.43a. From the cosine rule we have:
(1) $AB^2 = BO^2 + AO^2 - 2BO.AO\cos(\angle AOB)$
(2) $AD^2 = OD^2 + AO^2 + 2OD.AO\cos(\angle AOB)$

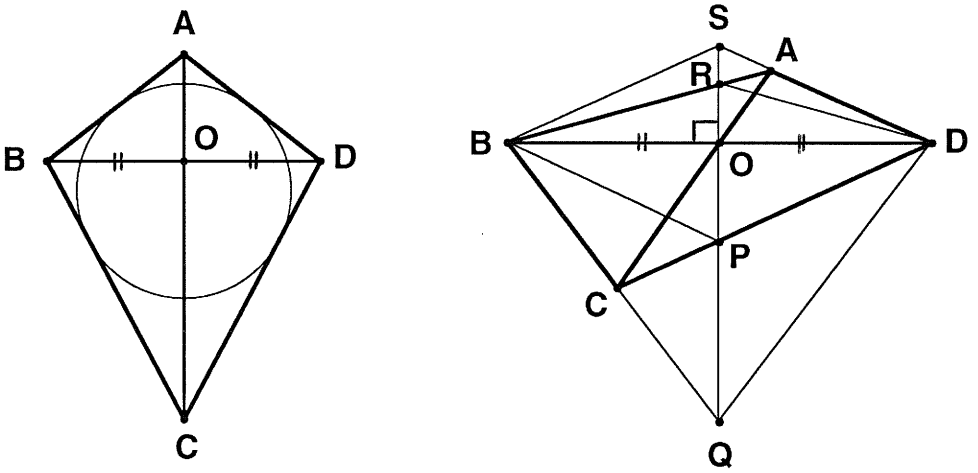


Figure 4.43

By addition, and from the equality of BO and OD, we obtain:
(3) $AB^2 + AD^2 = 2BO^2 + 2AO^2$.

Similarly, from triangles BOC and DOC we obtain:
(4) $BC^2 + CD^2 = 2BO^2 + 2OC^2$.

By subtracting (4) from (3) and re-arranging we obtain:
(5) $AB^2 + AD^2 - BC^2 - CD^2 - 2AO^2 + 2OC^2 = 0$.

Since ABCD is a circum quad, we also have $AB + CD = BC + AD$. Therefore,
(6) $AB = BC + AD - CD$.

Substitute (6) in (5) and simplify to obtain a quadratic equation in AD:
(7) $AD^2 + AD(BC - CD) - CD.BC - AO^2 + OC^2 = 0$.

Solving for AD and simplifying we obtain:

$$AD = \frac{(CD - BC) \pm \sqrt{(BC + CD)^2 + 4AO^2 - 4OC^2}}{2}.$$

Denote the discriminant by Δ to obtain:

$$AD = \frac{CD - BC + \sqrt{\Delta}}{2} \text{ or } AD = \frac{CD - BC - \sqrt{\Delta}}{2}.$$

If these values of AD are substituted into (6), we *correspondingly* obtain:

$$AB = \frac{BC - CD + \sqrt{\Delta}}{2} \text{ or } AB = \frac{BC - CD - \sqrt{\Delta}}{2}.$$

Carefully comparing the corresponding values, it should be clear that AD would only be equal to AD, if BC=CD. However, similarly solving for BC and CD we find that their equality in turn depends on the equality of AB and AD, which is what we want to prove! So here is a classic example of a circular argument.

Let's try a different approach by using *reductio ad absurdum.* Suppose AC is not perpendicular to BD as shown in Figure 4.43b. Construct $SOQ \perp BD$ with O the midpoint of BD and S and Q the respective intersections of the constructed perpendicular with DA extended and BC extended. Then from symmetry $\angle QBD = \angle QDB$ and $\angle PBD = \angle PDB$. Therefore $\angle CBD > \angle CDB$, and we have CD>BC ...(1). Similarly it follows that AB>AD ...(2). But it is given that ABCD is a circum quad, and therefore AB+CD=BC+AD ...(3). A substitution of (1) into (3) however leads to the conclusion that AB<AD, which clearly contradicts (2).

A similar contradiction can be obtained if AC is assumed to lie on the other side of the perpendicular SOQ. Therefore the assumption that AC is not perpendicular to BD must be false, and therefore AC is an axis of symmetry which implies that ABCD is a kite. (Does the same argument apply to the concave case?).

Personally the author usually only resorts to an indirect proof like that above when all direct approaches seem to fail. The problem is that an indirect proof seldom supplies any *explanation* why a result is true; it only shows that it cannot be false! However, it is very useful and in some cases probably more preferable than a long-winded explanatory proof, particularly if it is short and elegant.

7. (b) Yes, it is true as shown in Figure 4.44. This dual result can easily be proved by showing that the angle bisector of angle BOC and the perpendicular bisector of BC coincide and is left as an exercise to the reader.

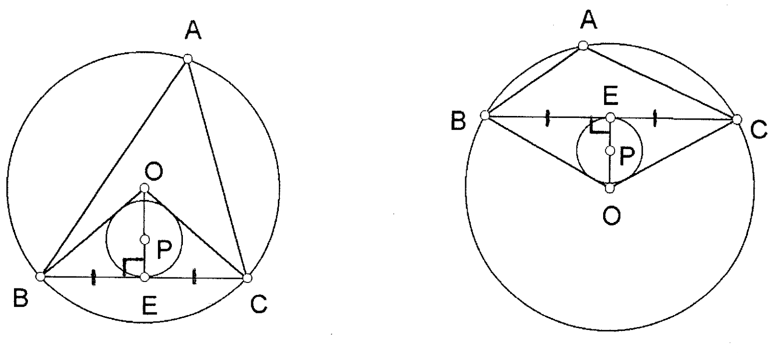


Figure 4.44

An interesting special case is when triangle ABC is isosceles, in which case the incentre O and circumcentre O' of triangle ABC are collinear with the incentre and circumcentre of the triangles BOC and BO'C (as well as with A). This result follows directly from symmetry as the angle bisector of angle A would then concide with the perpendicular bisector of BC.

12. It is always true. Consider Figure 4.22. Triangle AX_2X_5 is congruent to ΔX_1BX_4 ($\angle,\angle,s$) and P is the circumcentre of ΔAX_2X_5 and Q is the circumcentre of ΔX_1BX_4. In other words, a translation of A to X_1along segment AX_1 maps ΔAX_2X_5 onto ΔX_1BX_4, as well as P onto Q. Therefore $PQ//=AX_1$, but since $AX_1//=\frac{1}{3}AB$ we have $PQ//=\frac{1}{3}AB$. In a similar fashion can be shown that $PR//=\frac{1}{3}AC$ and $QR//=\frac{1}{3}BC$ from which follows the similarity of triangles ABC and PQR.

14. Both results are true and was discovered by the author during a review of *Cabri-geometre* in the journal **Pythagoras,** and the result where GHI is similar to the three outer triangles was then posed as a problem to the readers (see De Villiers, 1991b). Johan Meyer of the Department of Mathematics, University of the Orange Free State, Bloemfontein responded with the following elegant proof, which shows that the second conjectured dual (as well the original result), are merely special cases of a more general result (compare De Villiers, 1992).

In addition, it shows that we would also obtain similar triangles if we instead connected the corresponding *centroids* or *ortho-centres* of the similar triangles. This illustrates again the value of an explanatory proof which enables one to generalize a result by identifying the fundamental property upon which it depends.

Theorem 1
If similar triangles ADB, CBE and FAC are erected outwardly on the sides of any triangle ABC, their incentres G, H and I form a triangle similar to the three triangles.

Proof
We shall mainly use the following special case of the Petersen-Schoute theorem (Coxeter & Greitzer, 1967:99):

"If ABC and A'B'C' are two directly similar triangles, while AA'A", BB'B", CC'C" are three directly similar triangles, then ΔA"B"C" is directly similar to ΔABC (see Figure 4.45)."

Let us now first examine what we mean by points that are in the same relative position with respect to two directly similar triangles. (Two triangles are directly similar if the transformation which maps the one triangle on to the other preserves angles in both magnitude and direction).

Assume two triangles ABC and A'B'C' are directly similar (see Figure 4.46). Let P be any point in the plane. Then we can say that the point P' is in the same relative position to ΔA'B'C' as P is to ΔABC if the transformation (in this case a translation or a spiral similarity) which maps ΔABC to ΔA'B'C', also maps the point P onto P'.

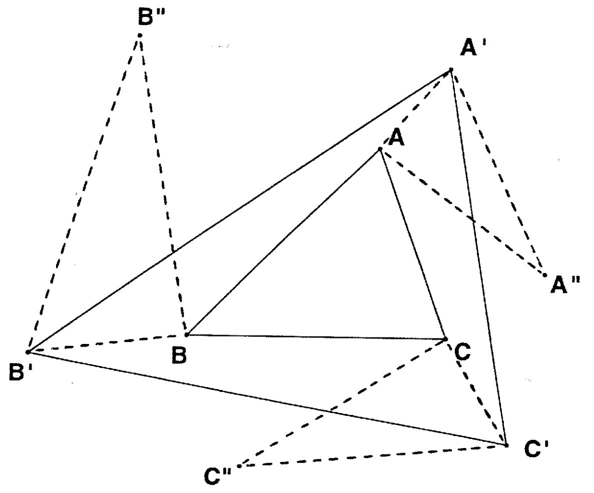


Figure 4.45

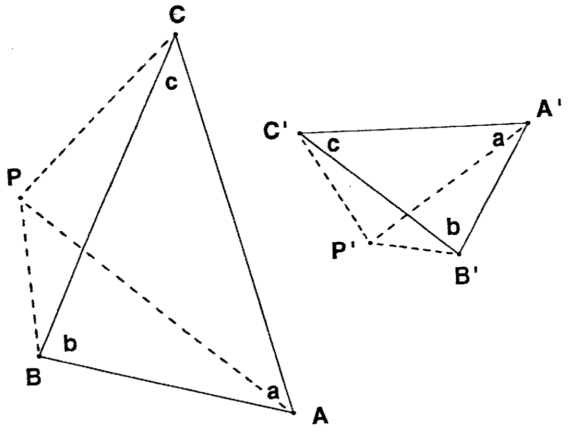


Figure 4.46

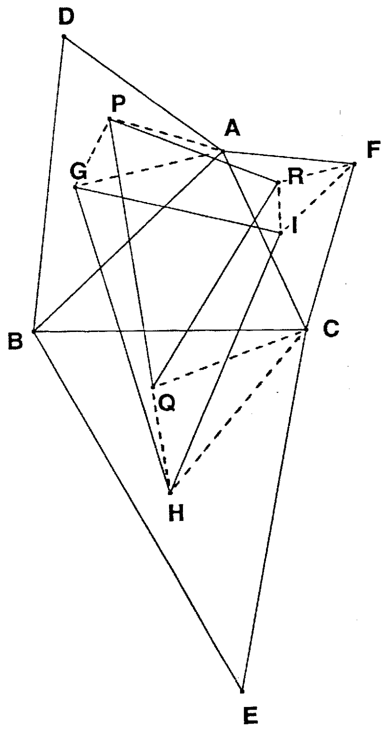


Figure 4.47

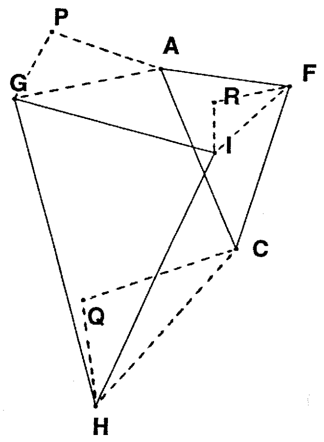


Figure 4.48

Let's now consider the above generalized dual to Napoleon's theorem. In Figure 4.47 we have ΔABC and the three directly similar triangles ADB, CBE and FAC. Now choose any three points P, Q and R in the plane so that they are respectively in the same relative positions to triangles ADB, CBE and FAC. (We could for example choose the three incentres). Let G, H and I be the three circumcentres of the similar triangles.

Then according to the generalization of Napoleon's theorem proved in *Questions & Problems 3*, no. 7, we have ΔGHI directly similar to the three similar triangles. We shall use the Petersen-Schoute theorem to show that ΔPQR is directly similar to ΔGHI which then provides the desired result.

Firstly we see that triangles PAG, QCH and RFI are directly similar since the points P, A and G have relatively the same positions to ΔADB as the points Q, C and H respectively have to ΔCBE. The points R, F and I similarly have the same relative positions to ΔFAC as the points Q, C and H have to ΔCBE. In Figure 4.48 we therefore have the same configuration as that of Figure 4.45 and the result now follows directly from the Petersen-Schoute theorem. (Note that triangle A'B'C' in Figure 4.45 does not have to include triangle ABC). It should also be observed that $\angle QPR = \angle BDA$, $\angle PQR = \angle BEC$ and $\angle QRP = \angle CFA$.

An interesting corollary is that we would also obtain a similar triangle if we instead connected the corresponding centroids or orthocentres of the similar triangles. The aforementioned generalization of Napoleon's theorem, as well as its dual, is therefore merely a special case of the following more general result:

"If similar triangles ADB, CBE and FAC are erected outwardly on the sides of any triangle ABC, and any three points P, Q and R are chosen so that they respectively lie in the same relative positions to these triangles, then P, Q and R form a triangle similar to the three triangles."

Theorem 2

If similar triangles ABD, EBC and AFC are erected outwardly on the sides of any triangle ABC, their incentres G, H and I form a triangle with $\angle G = \frac{1}{2}(\angle DAB + \angle DBA)$, $\angle H = \frac{1}{2}(\angle EBC + \angle ECB)$ and $\angle I = \frac{1}{2}(\angle FCA + \angle FAC)$.

Let us first prove the following lemma:

Lemma

Consider a triangle ABC with angles a, b and c and incentre O as indicated in Figure 4.49. If we construct a new triangle DBA with $\angle DBO = \frac{1}{2}c$ and $\angle DAO = \frac{1}{2}c$, then $\angle BDO = \frac{1}{2}a$ and $\angle ADO = \frac{1}{2}b$.

Proof

Extend BO and AO to E and F on AD and BD respectively. Then EFBA is a cylic quadrilateral, since $\angle FBE = \angle FAE = \frac{1}{2}c$. Therefore $\angle EFA = \angle EBA = \frac{1}{2}b$. But OEDF is also cyclic, since $\angle FOE + \angle FDE = 180°$. Therefore $\angle ODE = \angle OFE = \frac{1}{2}b$. Since $\angle BDA = \frac{1}{2}(a+b)$ it follows that $\angle ODF = \frac{1}{2}a$.

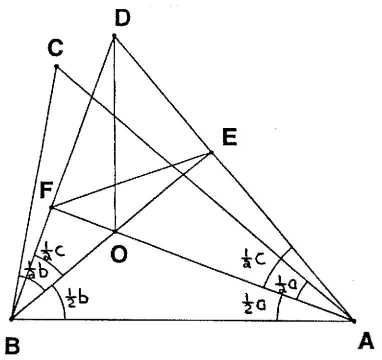


Figure 4.49

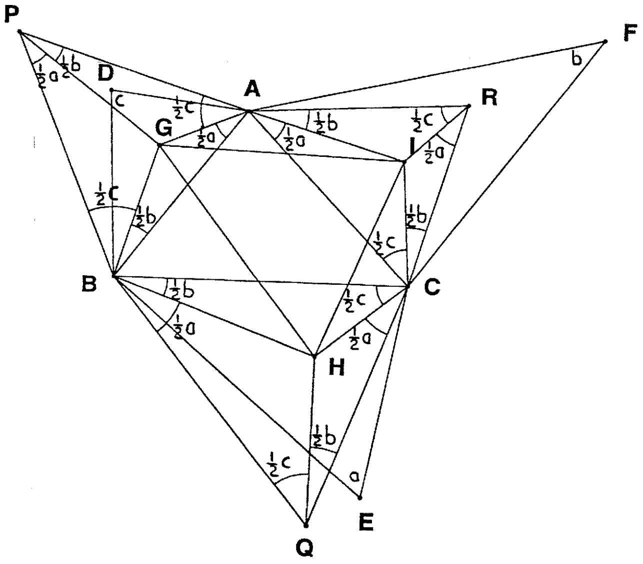


Figure 4.50

Proof of Theorem 2

In Figure 4.50 we have triangle ABC with three directly similar triangles ABD, EBC and AFC with angles *a*, *b* and *c* as indicated. Let G, H and I indicate the respective incentres.

Construct triangles ABP, CQB and RCA in the same way as in the Lemma above, in other words enlarge $\angle GAB$ and $\angle GBA$ each by $\frac{1}{2}c$ to obtain the point P, enlarge $\angle HBC$ and $\angle HCB$ each by $\frac{1}{2}a$ to obtain Q and enlarge $\angle IAC$ and $\angle ICA$ each by $\frac{1}{2}b$ to obtain R. These three new triangles now stand on the sides of triangle ABC in the same way as in the generalization of Theorem 1, with the angles P, Q and R respectively equal to $\frac{1}{2}(a+b)$, $\frac{1}{2}(b+c)$ and $\frac{1}{2}(a+c)$.

In addition, according to the above Lemma, we have that point G lies in the same relative position to ΔABP, as the point H lies relatively to ΔCQB. The same is true for the point I relative to ΔRCA. According to Theorem 1, we therefore obtain the desired result that $\angle G = \frac{1}{2}(a+b)$, $\angle H = \frac{1}{2}(b+c)$ and $\angle I = \frac{1}{2}(a+c)$.

We can now also easily prove the following interesting theorem:

Theorem 3

If triangles DBA, ECB and FAC with angles α, β and γ are erected outwardly on the sides of any triangle ABC as shown in Figure 4.51, and $\alpha+\beta+\gamma=90°$, then $\angle EDF = 2\beta$, $\angle DFE = 2\alpha$ and $\angle FED = 2\gamma$.

Proof

Construct three new triangles PBA, BQC and ACR by choosing $\angle PBD = \gamma$, $\angle PAD = \alpha$, $\angle QBE = \beta$, $\angle QCE = \alpha$, $\angle RCF = \gamma$ and $\angle RAF = \beta$. Then triangles

PBA, BQC and ACR are directly similar with incentres D, E and F respectively. Therefore ΔDEF is directly similar to ΔPBA so that $\angle EDF = \angle BPA = 2\beta$, etc.

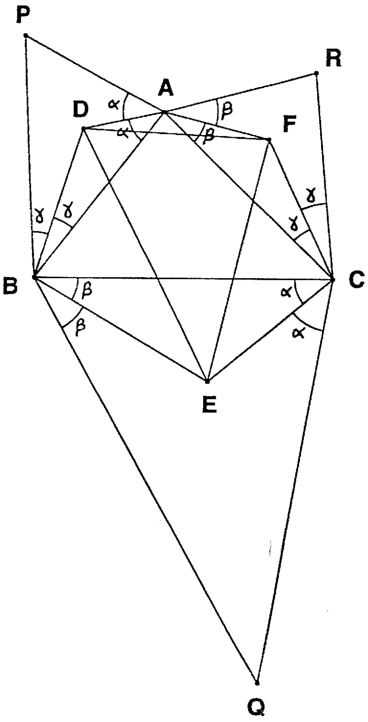


Figure 4.51

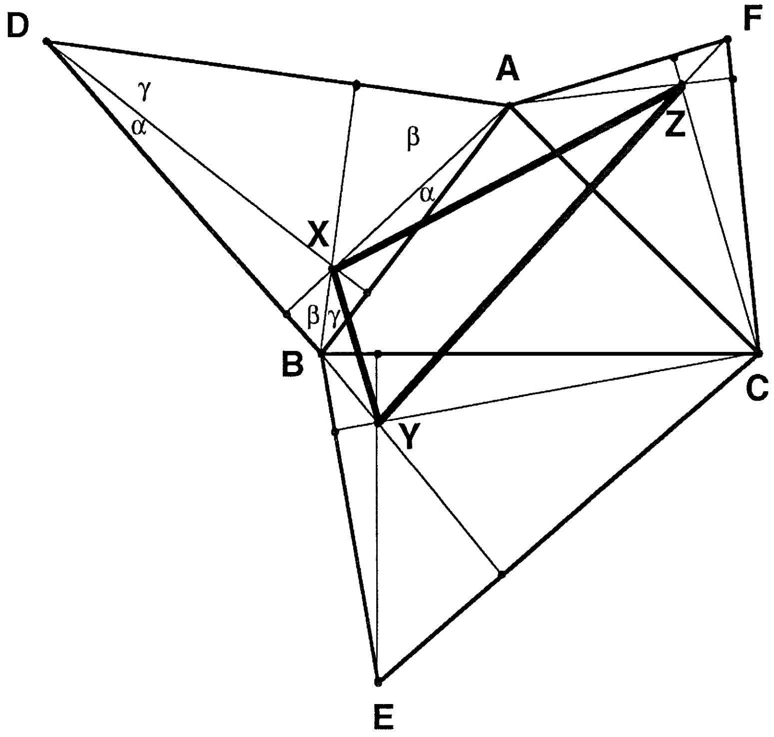


Figure 4.52

Furthermore, Theorem 3 has the following corollary:

If similar, acute-angled triangles DBA, CBE and CFA are erected outwardly on the sides of any triangle ABC as shown in Figure 4.52, then their orthocentres X, Y and Z with angles α, β and γ as shown, form a triangle with $\angle YXZ = 2\beta$, $\angle XZY = 2\alpha$ and $\angle XYZ = 2\gamma$.

What happens if the three similar triangles are right-angled or obtuse?

Lastly, can you formulate appropriate *converses* for Napoleon's theorem, its generalizations, extensions and duals? Investigate. (See Wetzel, 1992).

15. (b) As shown in Figure 4.53a the result can be generalized to similar rectangles on the sides, as it also fulfills the conditions of the Fermat generalization given in Figure 3.10. It is unfortunately not possible to construct similar rhombi on the sides of a triangle to fulfill those conditions (or in general those of Figure 3.12). It is also easy to see that this result is further generalizable to isosceles trapezia (see Figure 4.53b), and in general to any similar curvi-linear figures with axes of symmetry perpendicular to the sides and corresponding points (e.g. *centroids*) chosen on these lines of symmetry.

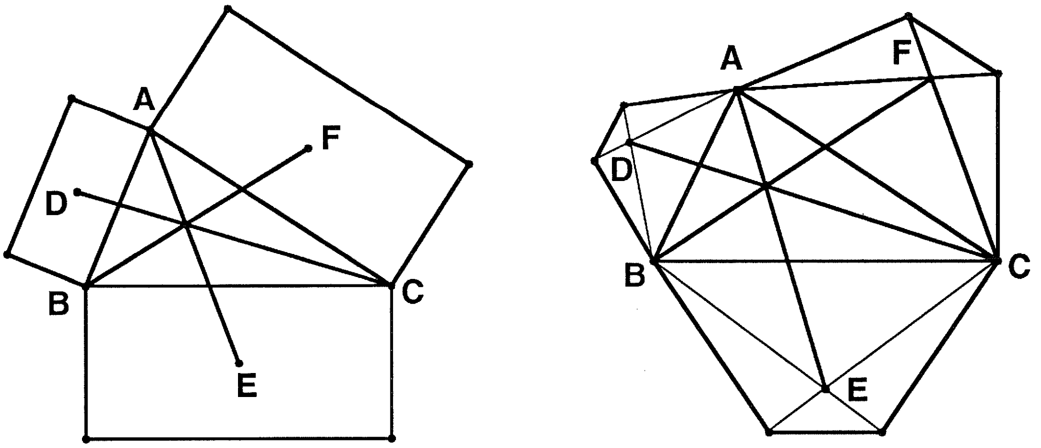


Figure 4.53

17. In an article in the **Mathematical Digest** Murray Klamkin (1991) pointed out that an analogous generalization exists for cyclic star *4n-gons*, namely:
(1) If $A_1A_2...A_{4n}$ $(n>1)$ is any cyclic star polygon P in which each vertex A_i is joined to vertex A_{i+2n-1}, then the two sums of alternate interior angles of P are each equal to π (see Figure 4.54a where $n = 2$).

However, as before, a dual also exists for this generalization (compare De Villiers, 1991):
(2) If $A_1A_2...A_{4n}$ $(n>1)$ is a circumscribed star polygon P in which each vertex A_i is joined to vertex A_{i+2n-1}, then the two sums of alternate sides are equal (see Figure 4.54b where $n = 2$).

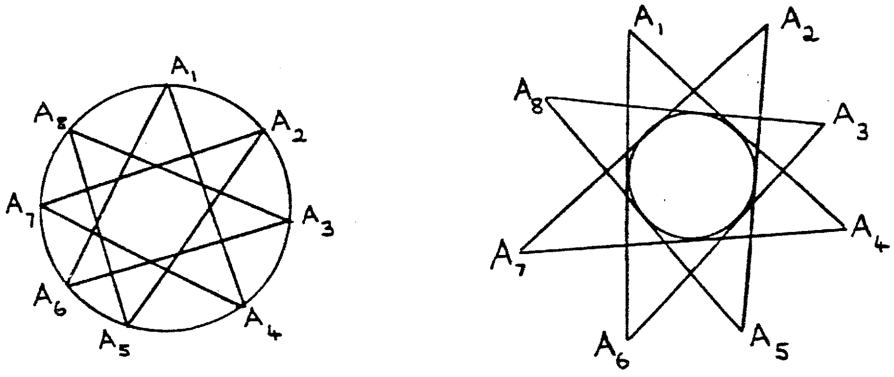


Figure 4.54

A unifying generalization

It is furthermore possible as shown in De Villiers (1993) to construct a class of cyclic $2n$-gons $(n>1)$ with the connecting rule $A_i \to A_{i+n-1}$ as shown in Figure 4.55 for which the total interior angle sum is 2π and the two sums of alternate angles are therefore equal to π. (Note that the cases n=4, 6, 8, 10... are the same polygons referred to in (1) above). The cases for n=3, 5, 7... can also be seen to consist of a set of two figures, i.e.

triangles (a generalized star of David), star pentagons, star septagons, etc., respectively overlapping in such a manner that A_1; A_3; A_5... A_{2n-1} belong to the one figure and A_2; A_4; A_6;... A_{2n} belong to the other figure. (Note therefore that in these cases the polygons need not be cyclic for the result to be true).

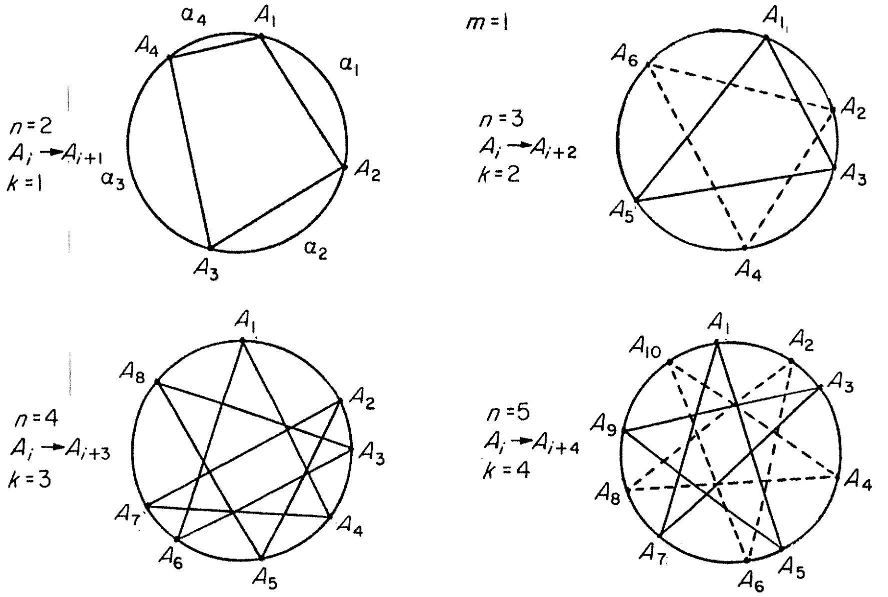


Figure 4.55

Next we can construct a class of cyclic $2n$-gons (n>2) with the connecting rule $A_i \rightarrow A_{i+n-2}$ as shown in Figure 4.56, for which the total interior angle sum is 4π and the two sums of alternate interior angles are therefore equal to 2π. Here the cases for n=4 and n=6 can also be seen to consist respectively of two overlapping quadrilaterals and two generalized stars of David, and therefore need not be cyclic.

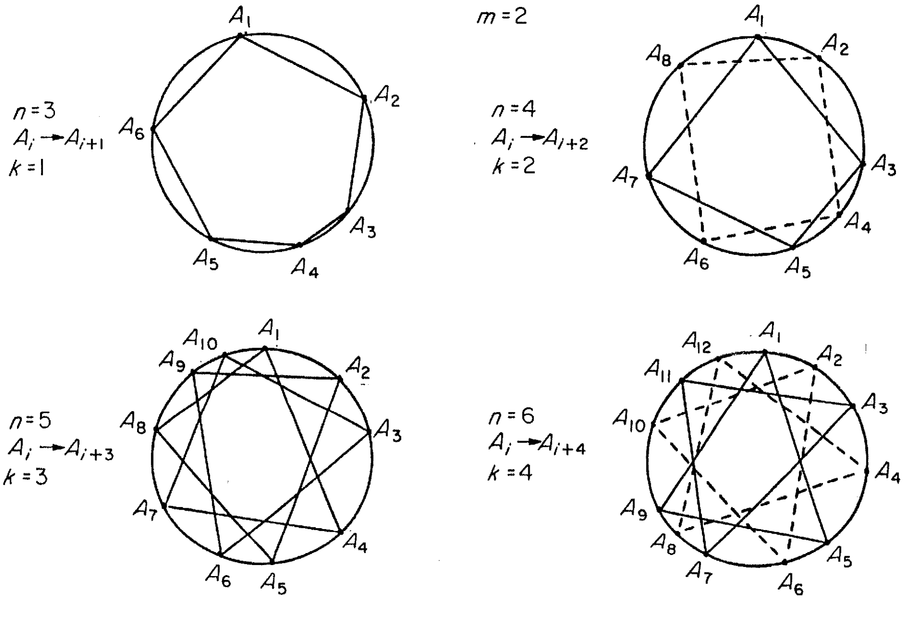


Figure 4.56

In a similar fashion other general classes can also be formed from convex cyclic octagons, decagons, etc. The connecting rule would then in general simply be $A_i \to A_{i+n-m}$ with $m = 1, 2, 3, \ldots$ and $n - m \geq 1$. Or alternatively and more simply, the general connecting rule can be formulated as $A_i \to A_{i+k}$ where $k = 1, 2, 3, \ldots$ is the **total turning** (the number of complete rotations of 2π) one would undergo walking completely around the perimeter of each figure. We can therefore now formulate the following beautiful generalization of Turnbull's theorem which includes all the previous cases:

Theorem 1

If $A_1A_2 \ldots A_{2n}$ $(n > 1)$ is any cyclic $2n$-gon in which vertex $A_i \to A_{i+k}$ (vertex A_i is joined to A_{i+k}), then the two sums of alternate interior angles are each equal to $m\pi$ (where $m = n - k$).

Although generalization (1) given earlier can easily be proved by mathematical induction from the special case for cyclic quadrilaterals and considering the addition of two vertices at a time, the following proof, based on the notation and approach of Klamkin (1991), turned out more convenient for the unifying generalization.

Proof

Let the measure of the minor arc $A_iA_{i+1} = \alpha_i$; $i = 1;2;\ldots 2n$ with $\alpha_{2n+1} = \alpha_1$. (See first figure in Figure 4.55). First consider the case for $m = 1$ and the sum of the odd alternate angles, then:

$$\begin{aligned} 2\angle A_1 &= \alpha_n + \alpha_{n+1} \\ 2\angle A_3 &= \alpha_{n+2} + \alpha_{n+3} \\ \vdots \quad &\quad \vdots \\ 2\angle A_{2n-1} &= \alpha_{n-2} + \alpha_{n-1} \end{aligned}$$

Then by addition we have: $2\sum_{i=1}^{n} \angle A_{2i-1} = 2\pi \Leftrightarrow \sum_{i=1}^{n} \angle A_{2i-1} = \pi$.

In the same manner we can prove that the sum of the even alternate interior angles is also π. (Or alternatively we can in general find the sum of the even alternate angles simply by subtracting πm from the total interior angle sum of these $2n$-gons, namely $S = 2\pi m$. This follows directly as a special case from the general proof for n-gons given in *Solutions 3*, no. 15 (compare De Villiers, 1989)).

After also considering the case for $m = 2$, the general argument is easy to formulate as follows:

$$\begin{aligned} 2\angle A_1 &= \alpha_{n-m+1} + \alpha_{n-m+2} + \ldots + \alpha_{n+m-1} + \alpha_{n+m} \\ 2\angle A_3 &= \alpha_{n-m+3} + \alpha_{n-m+4} + \ldots + \alpha_{n+m+1} + \alpha_{n+m+2} \\ \vdots \quad &\quad \vdots \\ 2\angle A_{2n-1} &= \alpha_{n-m-1} + \alpha_{n-m} + \ldots + \alpha_{n+m-3} + \alpha_{n+m-2} \end{aligned}$$

Then by addition and rearrangement we have:

$$2\sum_{i=1}^{n}\angle A_{2i-1} = m(\alpha_{n+m} + \alpha_{n+m+1} + \ldots + \alpha_{n+m-2} + \alpha_{n+m-1}) = m(2\pi)$$

$$\Leftrightarrow \ \sum_{i=1}^{n}\angle A_{2i-1} = m\pi$$

Or alternatively, mathematical induction can be used in an informative manner as follows. As it is true for $m = 1$ as already shown, let's assume that it is true for $m = p$, and therefore that the following is true:

$$\begin{aligned}
2\angle A_1 &= \alpha_{n-p+1} + \alpha_{n-p+2} + \ldots + \alpha_{n+p-1} + \alpha_{n+p} \\
2\angle A_3 &= \alpha_{n-p+3} + \alpha_{n-p+4} + \ldots + \alpha_{n+p+1} + \alpha_{n+p+2} \\
\vdots \quad &\quad \vdots \\
2\angle A_{2n-1} &= \alpha_{n-p-1} + \alpha_{n-p} + \ldots + \alpha_{n+p-3} + \alpha_{n+p-2}
\end{aligned}$$

$$\Rightarrow \ 2\sum_{i=1}^{n}\angle A_{2i-1} = p(2\pi) \ \Leftrightarrow \ \sum_{i=1}^{n}\angle A_{2i-1} = p\pi.$$

Now consider $m = p+1$, then:

$$\begin{aligned}
2\angle A_1 &= \alpha_{n-p} + \quad \alpha_{n-p+1} + \ldots + \alpha_{n+p} \quad + \alpha_{n+p+1} \\
2\angle A_3 &= \alpha_{n-p+2} + \quad \alpha_{n-p+3} + \ldots + \alpha_{n+p+2} \quad + \alpha_{n+p+3} \\
\vdots \quad &\quad \vdots \\
2\angle A_{2n-1} &= \alpha_{n-p-2} + \quad \alpha_{n-p-1} + \ldots + \alpha_{n+p-2} \quad + \alpha_{n+p-1}
\end{aligned}$$

By addition, rearrangement and utilization of the assumed truth for $m = p$ we then obtain:

$$2\sum_{i=1}^{n}\angle A_{2i-1} = p(2\pi) + 2\pi \ \Leftrightarrow \ \sum_{i=1}^{n}\angle A_{2i-1} = (p+1)\pi.$$

This shows that it is true for $m = p+1$, but the argument is true for $m = 1$, and therefore according to the principle of mathematical induction, it would be true for all $m =$ 1, 2, 3, ...

Furthermore, in accordance with generalizations (1) and (2) above, the above unifying generalization of Turnbull's theorem has the following interesting dual:

Theorem 2

If $A_1A_2\ldots A_{2n}$ $(n > 1)$ is any circumscribed $2n$-gon in which vertex $A_i \to A_{i+k}$, then the two sums of alternate sides are equal.

In Figure 4.57, the first four dual examples are given for m=1. Note that in theorem 1 the sum of the alternate angles is equal to half the total angle sum, while in the dual result we have the sum of the alternate sides equal to half the perimeter. However, in contrast to Theorem 1, the two sums of alternate sides are not constant for the same value of m, as the perimeter of a circumscribed $2n$-gon varies, not only with respect to the size of the circle, but also for a circle with a fixed radius. Furthermore, it should be noted that it is necessary here to give a more general meaning to the concept "*side*". For example, for $n = 3$ the "*sides*" are to be interpreted respectively as $A_1B_1 + A_2B_6$, etc. In other words, a "*side*" is the sum of the intersecting tangents drawn from adjacent vertices. As before, the proof of Theorem 2 is based on the theorem that the tangents drawn from a point outside a circle are equal, and is left to the reader.

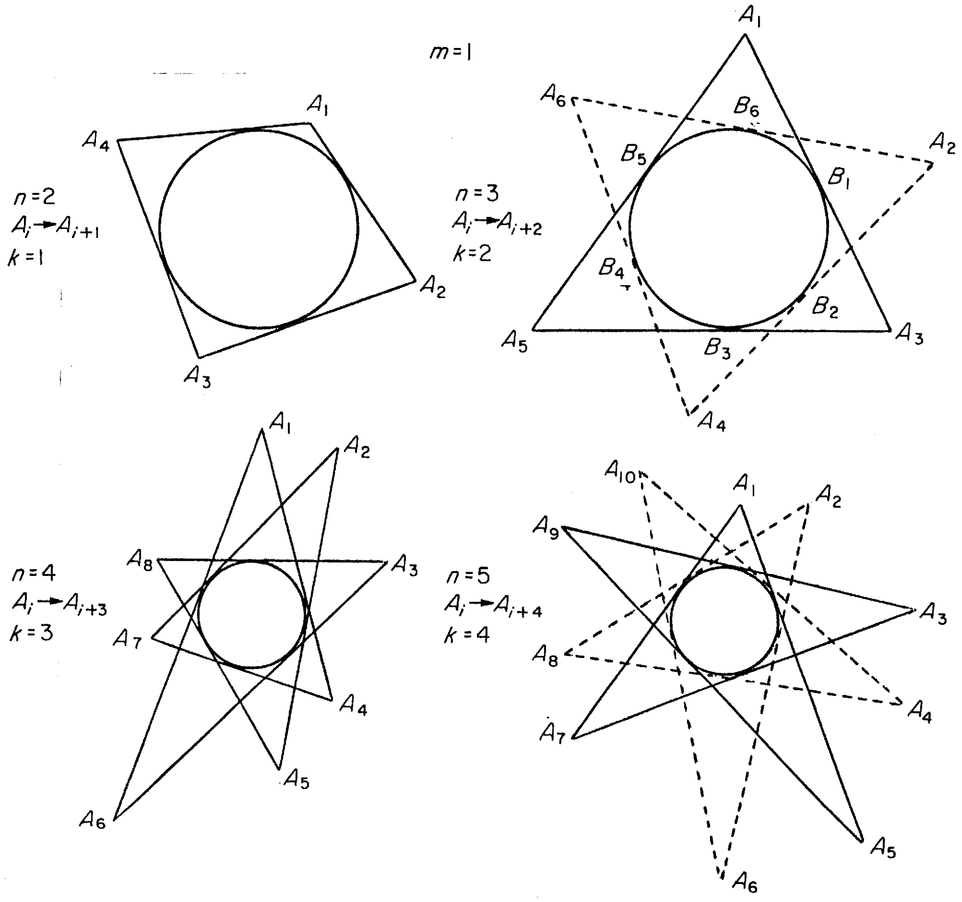


Figure 4.57

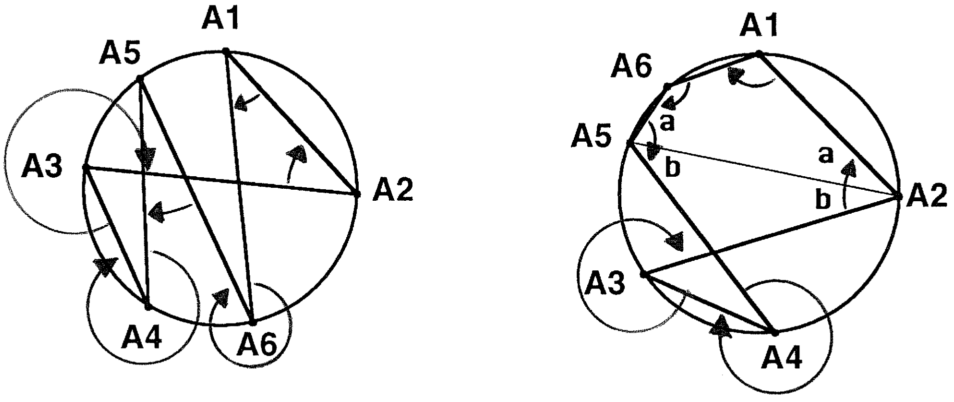


Figure 4.58

Further extensions?

What happens if we have crossed cylic hexagons as shown in Figure 4.58? In the first case, we do not have the sums of alternate angles equal since $\angle A_1 + \angle A_3 + \angle A_5 = 2\pi$ and $\angle A_2 + \angle A_4 + \angle A_6 = 4\pi$ (the proof is left to the reader). However, in the second case the result holds.

For example, draw A_2A_5 then we have $\angle A_1 + \angle A_{5a} = \angle A_{2a} + \angle A_6 = \pi$ and $\angle A_{5b} + \angle A_3 = \angle A_{2b} + \angle A_4 = 2\pi$. By addition we therefore obtain the desired result, namely: $\angle A_1 + \angle A_3 + \angle A_5 = \angle A_2 + \angle A_4 + \angle A_6 = 3\pi$.

Can you further generalize this result? Is this result included in the previous generalization or can the generalization be reformulated to include it? Can you formulate a dual result? Is the dual result true or not? Investigate.

18. (b) It is easy to see that the similarity of triangles AOE and COF are maintained if we construct similar rectangles or rhombi as shown in Figure 4.59a-b. In fact, we can easily generalize it further to similar isosceles trapezia or kites as shown in Figure 4.59c-d. Can you generalize it still further?

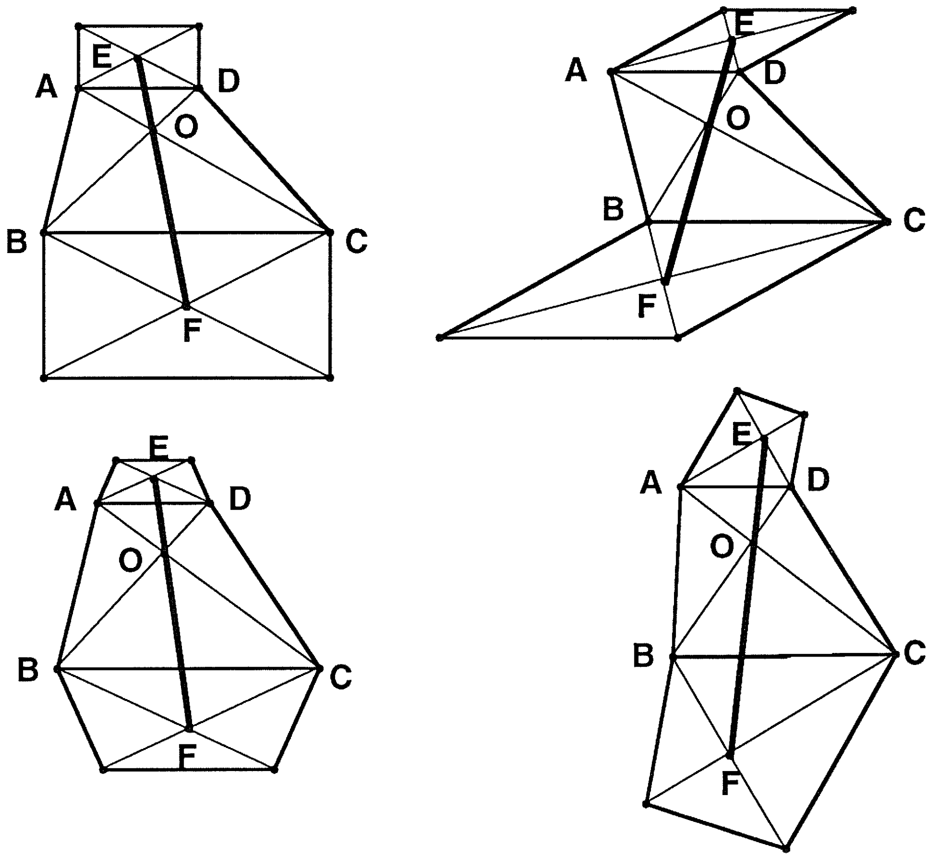


Figure 59

20. (c) This very beautiful result is known as Von Aubel's theorem and is true for convex, concave or crossed quadrilaterals and the squares can be constructed outward or inward. A general vector proof is given in Kelly (1966).

Note that the line segments joining the centres of opposite squares (the diagonals of EFGH) need not intersect, but nevertheless remain equal and perpendicular. The result can be *specialized* in a number of different ways, for example:

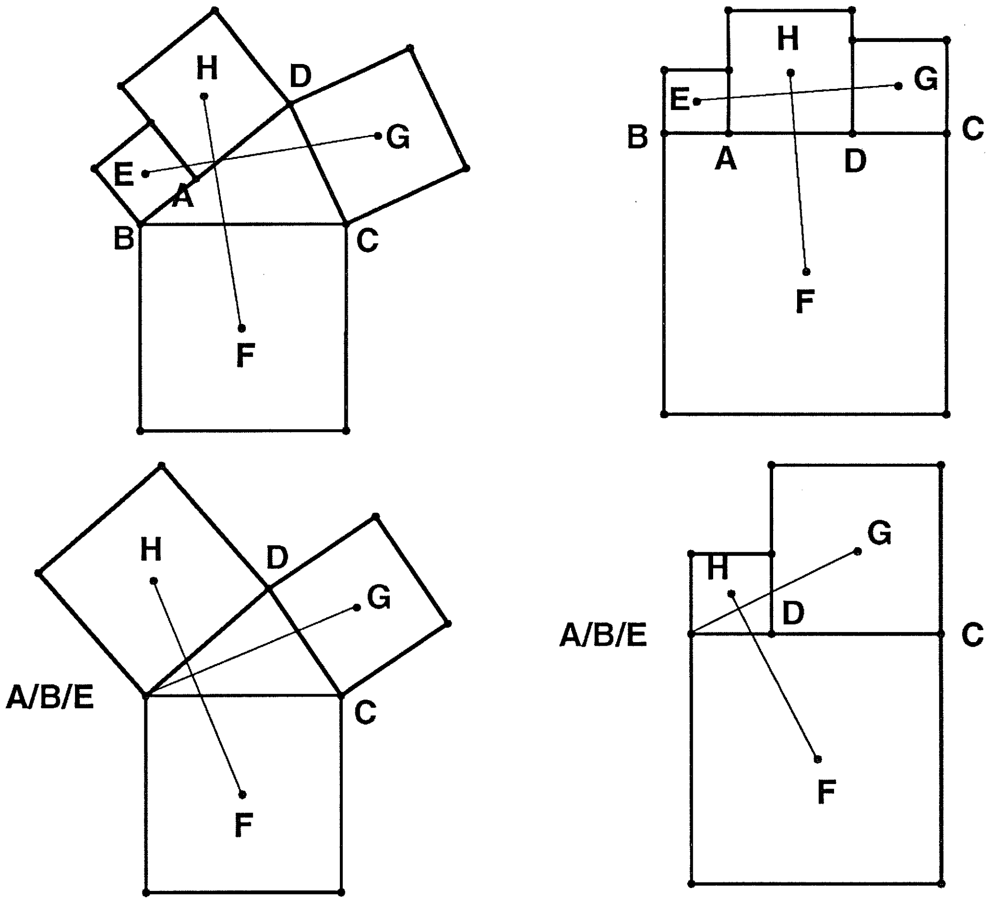


Figure 4.60

(1) Any three or even all four of the quadrilateral's corners may be collinear. In the first case the quadrilateral degenerates into a triangle with a "*vertex*" on one side (see Figure 4.60a); in the second case into a straight line with two "*vertices*" on it (see Figure 4.60b).

(2) One side of the quadrilateral may have zero length if we let two vertices coincide as shown in Figures 4.60c-d. If we further let C and D coincide in Figures 4.60c-d then it is easy to see that the line segment joining the centres of the two squares drawn on opposite sides of the resulting line segment joining A/B/E to C/D would be perpendicular and equal to it.

What happens if instead of squares we construct similar rectangles or rhombi on the sides? Investigate. (See Chapter 8).

21. (i) The results in (a) and (b) can be generalized as follows and explains why the previous results are dual:

"The respective centroids P, Q, R and S of triangles ABC, BCD, CDA and DAB of any quadrilateral ABCD form a quadrilateral QRSP *similar* to the original".

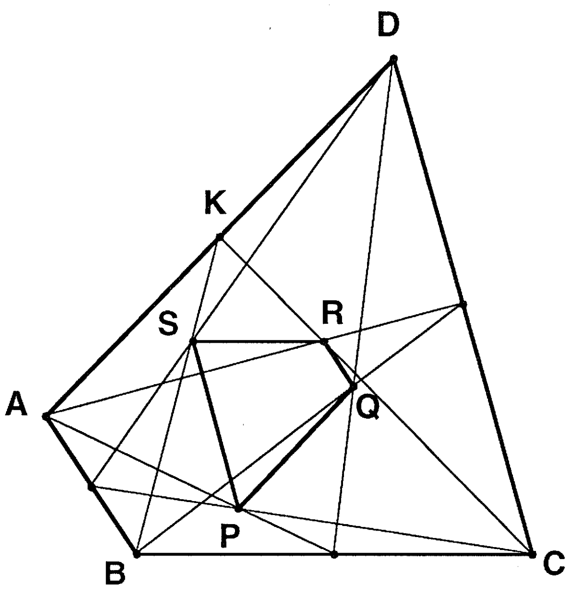


Figure 4.61

Proof

Consider Figure 4.61. Since S and R are the respective centroids of triangles DAB and CDA, we have KS/SB=1/3 and KR/RC=1/3. Therefore in ΔKBC, we see that RS//=⅓BC. Similarly, it can be shown that the other sides of PQRS are parallel to the respective sides of the original quadrilateral DABC and proportional to these sides, and thus completes the proof. (Also note the concurrency of the lines AQ, BR, CS and DB at the centre of the similarity which maps ABCD to QRSP).

(ii) If we connect the incentres of triangles ABC, BCD, CDA and DAB of an isosceles trapezium, we obtain a rectangle as shown in Figure 4.62a. This result can be generalized to any cyclic quadrilateral.

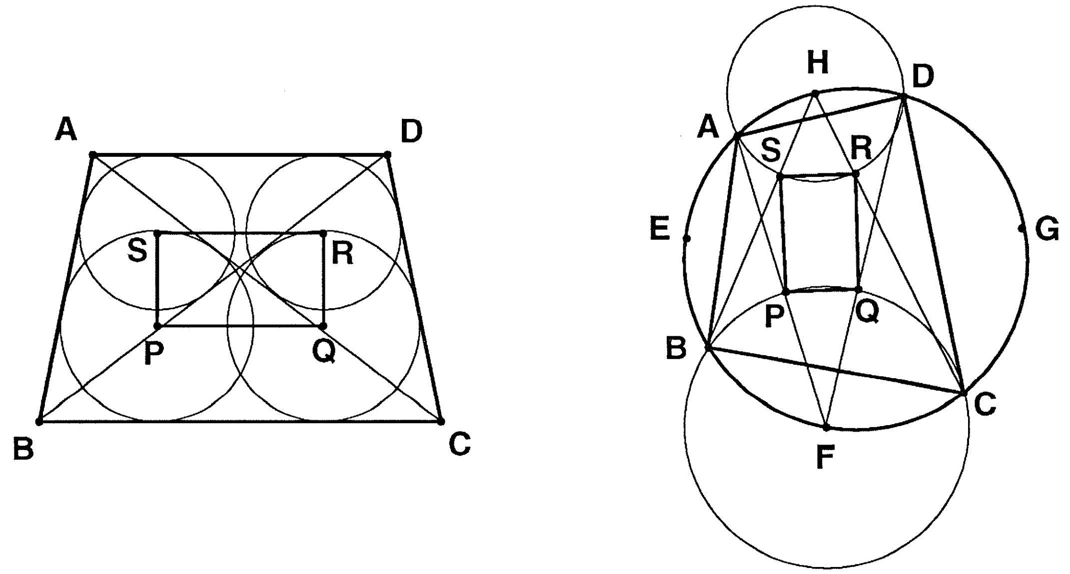


Figure 4.62

Proof

Consider Figure 4.62b. Let E, F, G and H be the midpoints of the arcs subtended by the sides AB, BC, CD and DA of the cyclic quadrilateral ABCD on the circumcircle. The incentres P and Q of the triangles ABC and BCD lie on the angle bisectors AF and DF of the angles BAC and BDC and on the circle with F as centre and FB as radius (see Lemma 1 below); thus the triangle FPQ is isosceles, and the base PQ is perpendicular to the bisector FH of $\angle F$. Similarly the line FH is perpendicular to RS joining the incentres R and S of triangles CDA and DAB.

In the same way, the line EG is perpendicular to QR and AS. But the lines FH and EG are perpendicular (see Lemma 2); hence the result.

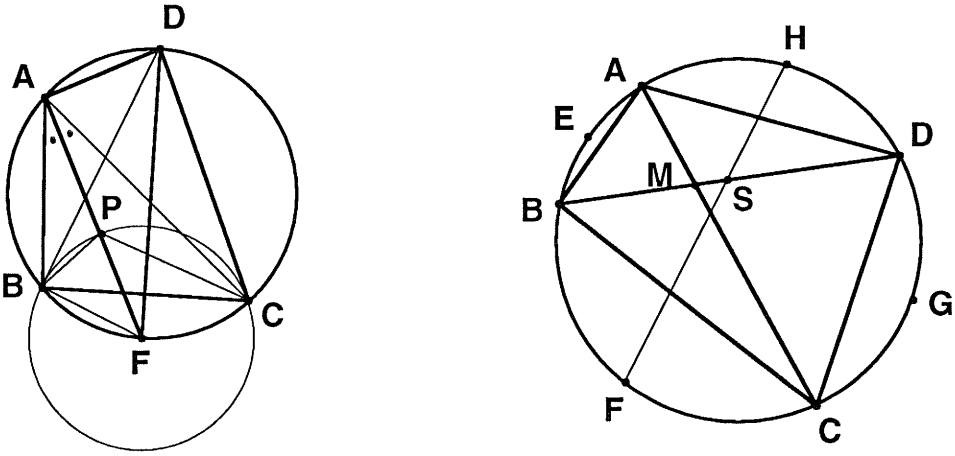


Figure 4.63

Lemma 1

Consider Figure 4.63a with cyclic quadrilateral ABCD. Construct the angle bisector AF of $\angle BAC$ with F on the circle. Connect F with D, and F with B. Now we have $\angle BAC = \angle BDC$ on chord BC, but $\angle BAF = \angle BDF$ on chord BF; therefore DF is the angle bisector of $\angle BDC$. Furthermore, since $\angle BAF = \angle FDC$, we have BF=FC and therefore F is the midpoint of arc BC.

Construct a circle with F as centre and BF as radius, and label its intersection with AF as P. Connect P with C. We now have $\angle AFB = \angle ACB$ on chord AB, but $\angle PFB (= \angle AFB) = 2\angle PCB$ on chord BP; therefore CP is the angle bisector of $\angle ACB$ and P is the incentre of ΔABC.

Lemma 2

Consider Figure 4.63b with ABCD a cyclic quadrilateral and E, F, G and H the midpoints of the arcs as indicated.

The angle AMD is measured by half the sum of the arcs AD and BC. Now HF meets the diagonal BD in a point, S, inside the circle; hence angle HSD is measured by half the sum of the arcs HD and BF, and is therefore equal to half of the angle AMD, i.e. the line FH is parallel to the angle bisector of angle AMD. Similarly, EG is parallel to the angle bisector of angle AMB; hence FH and EG are perpendicular.

(Is this result, and argument, valid for a crossed cyclic quadrilateral? Check.)

(iii) If we connect the incentres (or circumcentres) of triangles ABC, BCD, CDA and DAB of a kite it is easy to see that we unfortunately do not necessarily obtain a rhombus; so the aforementioned result does not seem to have a dual.

However, if we connect the respective circumcentres of triangles ABC, BCD, CDA and DAB of a circum quad ABCD as shown in Figure 5.64, then PQRS is also a circum quad. By using dynamic software like *Sketchpad* or the property checker of *Cabri*, it can easily be verified that the angle bisectors of the angles of PQRS remain concurrent.

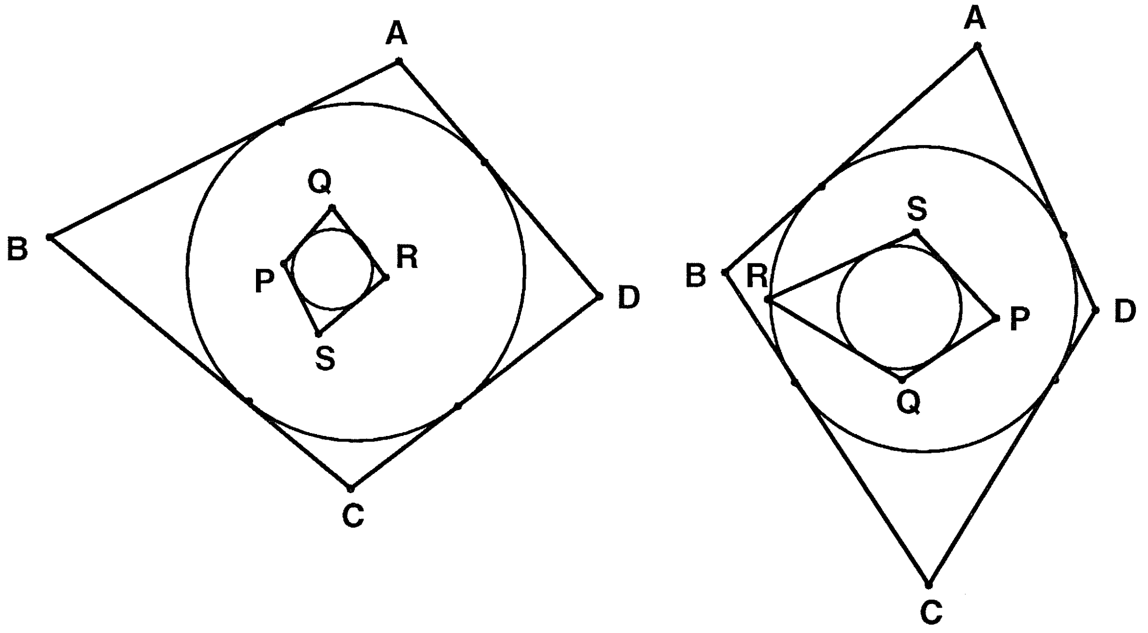


Figure 4.64

Can you explain why this result is true? (See no. 23b).

23. (a)(i) The result is true for crossed quadrilaterals as shown in Figure 4.65. For example, acute angle FAD$=180°-\frac{1}{2}\angle A$ and acute angle ADF$=180°-\frac{1}{2}\angle D$. Therefore $\angle EFG = 360° - \frac{1}{2}\angle A - \frac{1}{2}\angle D$. Similarly, $\angle EHG = 360° - \frac{1}{2}\angle B - \frac{1}{2}\angle C$. Thus $\angle EFG + \angle EHG = 720° - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 360°$ from which follows that the crossed quadrilateral EFGH is cyclic.

(ii) The external angle bisectors of a convex quadrilateral also form a cyclic quadrilateral, but the proof is left to the reader. (Do the external angle bisectors of a crossed quadrilateral also form a cyclic quadrilateral?)

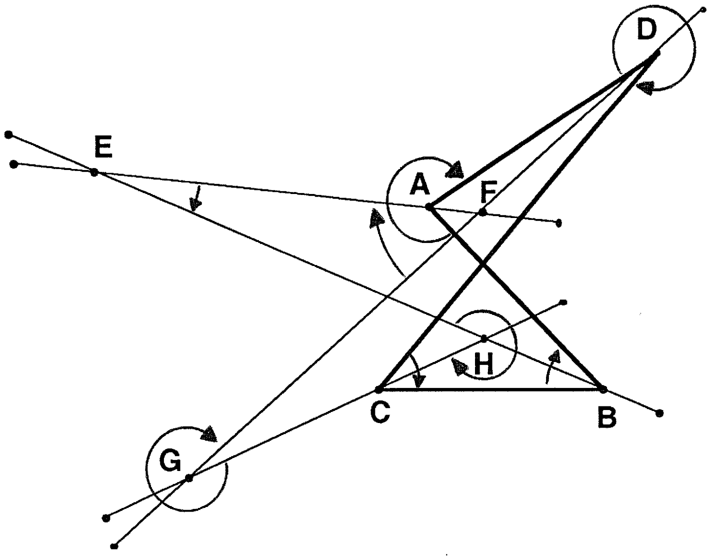


Figure 4.65

(b) After some unsuccessful attempts to (geometrically) prove this result, the author wrote to several mathematicians requesting their assistance in proving it. The author is indebted to Jordan Tabov, Institute of Mathematics, Bulgarian Academy of Sciences for the proof given below, as well as a similar one from John Wilker, Dept. of Mathematics, University of Toronto. As far as the author knows this result is original and has not been published elsewhere before. (The well-known geometer H.S.M. Coxeter has also admitted not seeing this result before).

Proof

We shall use the notation given in Figure 4.66 and also the following: AB=a, BC=b, CD=c, DA=d, t_A is the tangent from A to the incircle, r is the radius of the incircle. We shall assume that r=1. We now have:

$$MD = -\frac{d}{2\cos D},\quad MW = \frac{c}{2} - \frac{d}{2\cos D},$$

$$FW = MW\cot(180° - D) = \frac{d}{2\sin D} - \frac{c}{2}\cot D.$$

Similarly, $FX = \dfrac{c}{2\sin D} - \dfrac{d}{2}\cot D$ and hence:

$$\begin{aligned}2(FX - FW) &= \frac{c-d}{\sin D} + (c-d)\cot D\\ &= (c-d)\frac{1+\cos D}{\sin D}\\ &= (c-d)\cot\frac{D}{2}\\ &= (c-d)\frac{t_D}{r}\\ &= (c-d)t_D.\end{aligned}$$

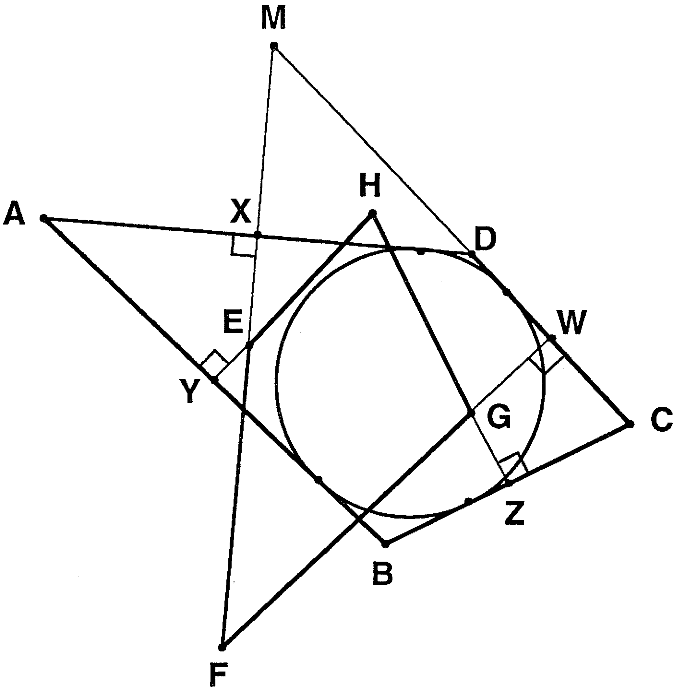


Figure 4.66

Similarly for 2(HZ-HY), 2(EY-EX) and 2(GW-GZ). We now have to check that the sum of the opposite sides are equal, or alternatively that EF-EH+HG-GF=0, i.e. that (FX-EX)-(HY-EY)+(HZ-GZ)-(FW-GW)=0 or 2(FX-FW)+2(HZ-HY)+2(EY-EX)+ 2(GW-GZ)=0. In view of the previous results, the last equality is equivalent to:
(1)... $(c-d)t_D+(d-a)t_A+(a-b)t_B+(b-c)t_C=0$.

Since $c-d=b-a$ and $d-a=c-b$, equation (1) is equivalent to:
(2)... $(c-d)(t_D-t_B)+(d-a)(t_A-t_C)=0$.

But $c-d=(t_C+t_D)-(t_D+t_A)=t_C-t_A$ and similarly $d-a=t_D-t_B$. Therefore (2) is equivalent to $(t_C-t_A)(t_D-t_B)+(t_D-t_B)(t_A-t_C)=0$, which is an identity and completes the proof.

Is this result also true for a concave circum quad? If so, is the above proof valid as given or can you adapt it? If not, can you provide a counter-example?

An interesting related result

In a recent article Branko Grunbaum (1993) from the Dept. of Mathematics, University of Washington proved the following related result by using *Mathematica*: "The perpendicular bisectors of the sides of a quadrilateral Q form a quadrilateral Q_1, and the perpendicular bisectors of the sides of Q_1 form a quadrilateral Q_2 which is similar to Q".

In relation to our result above, this implies that if we construct the perpendicular bisectors of the sides of the circum quad EFGH we obtain another circum quad *similar* to the original circum quad ABCD. Grunbaum's use of computer technology for the verification of this result is also highly significant. As he points out in his article: "*Do we start trusting numerical evidence (or other evidence produced by computers) as proofs of mathematics theorems?... if we have no doubt - do we call it a theorem?...I do think that my assertions are theorems...the mathematical community needs to come to grips with new modes of investigation that have been opened up by computers*".

Can you formulate a dual to Grunbaum's result? Investigate whether it is true or not.

Another interesting property of the configuration shown in Figure 4.66 is that the angle bisectors of EFGH are respectively parallel to the angle bisectors of ADCB. (The proof is simple and is left to the reader). Can this property perhaps be used to construct a different proof to the one given above?

24. (a) & (b) The dual is easily confirmed with *Sketchpad* by moving Q along the circumcircle while measurements, and calculations on them, are continuously updated. Again it was this experimental confirmation that motivated the author to start looking for a proof (see De Villiers, 1995c). Both results will now be proved below with reference to Figure 4.67.

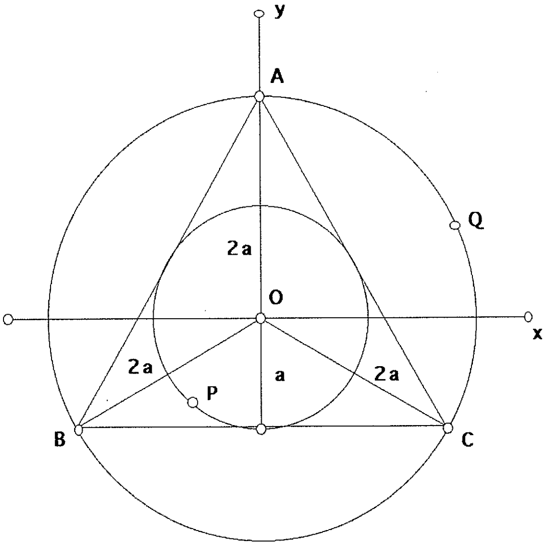


Figure 4.67

Proof of first result

Place the equilateral ΔABC with its incentre/circumcentre at (0; 0). Without loss of generality, let $A(0;\ 2a)$, $B(-\sqrt{3}a;\ -a)$, $C(\sqrt{3}a;\ -a)$ and $P(x;\ y)$. Then the incircle is $x^2 + y^2 = a^2$ and $PA^2 + PB^2 + PC^2$

$$=\left[x^2+(y-2a)^2\right]+\left[(x+\sqrt{3}a)^2+(y+a)^2\right]+\left[(x-\sqrt{3}a)^2+(y+a)^2\right]$$
$$=3(x^2+y^2)+12a^2$$
$$=3a^2+12a^2$$
$$=15a^2$$

Since a is constant, it follows that the sum of the squares of the distances from P to the vertices must also be constant.

Proof of dual

To prove the dual, one needs to know (or deduce) what the squared distance is from a point $(p;\ q)$ to a given line $y = mx + c$, which is given as:

$$\left(p-\frac{mq+p-cm}{m^2+1}\right)^2+\left(q-\frac{c+m^2p+mp}{m^2+1}\right)^2.$$

If we again consider the triangle in Figure 4.67, then the circumcircle is $x^2+y^2=4a^2$, AB is $y=\sqrt{3}x+2a$, AC is $y=-\sqrt{3}x+2a$ and BC is $y=-a$. Then using the above result we have $h_{AB}^2+h_{BC}^2+h_{AC}^2$

$$=\left(x-\frac{\sqrt{3}y+x-2\sqrt{3}a}{4}\right)^2+\left(y-\frac{2a+3y+\sqrt{3}x}{4}\right)^2$$
$$+\left(x-x\right)^2+\left(y-(-a)\right)^2+\left(x-\frac{-\sqrt{3}y+x+2\sqrt{3}a}{4}\right)^2$$
$$+\left(y-\frac{2a+3y-\sqrt{3}x}{4}\right)^2$$
$$=3a^2+\frac{3}{2}\left(x^2+y^2\right)$$
$$=3a^2+\frac{3}{2}\times 4a^2$$
$$=9a^2$$

Since a is constant, it follows that the sum of the squares of the distances from Q to the sides are constant. Furthermore, the ratio between these two constants can be seen to be 5/3. Note that the above proofs can be somewhat simplified by setting $a = 1$, although it is nice to obtain the explicit formulae in terms of a.

Can you generalize these two results further? Investigate. (See 24 (cont.) at end of this chapter).

25. (a) Both results are true for circum and cyclic hexagons as can be rather nicely illustrated and verified by *Sketchpad* or *Cabri*.

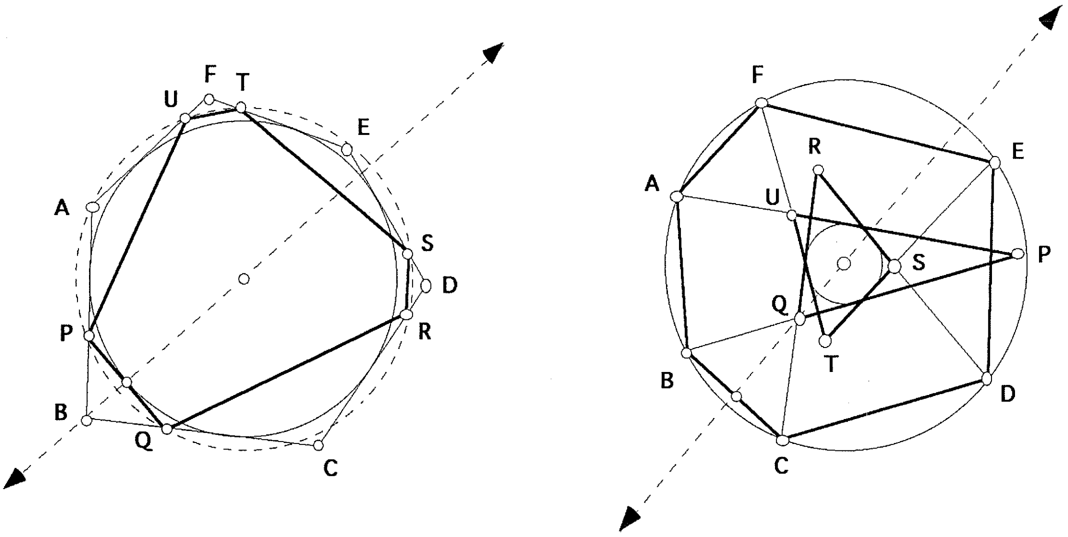


Figure 4.68

First result

Select any point P on AB of a circum hexagon ABCDEF. Take Q on BC so that BQ = PB, R on CD so that CR = QC, S on DE so that DS = RD, T on EF so that ET = SE and U on FA so that FU = TF. Then AU = AP and PQRSTU is a cyclic hexagon.

Proof

Consider Figure 4.68a. Denote AB by a, BC by b, etc. Let $PA = x$, then $PB = a-x = BQ$, $QC = b-a+x = CR$, $RD = c-b+a-x = DS$, $SE = d-c+b-a+x = ET$ and $TF = e-d+c-b+a-x = FU$. But the two sums of alternate sides are equal for a circum hexagon as shown in *Questions and Problems 4*, No. 17, ie. $a+c+e = b+d+f$. Therefore by substituting $a+c+e$ by $b+d+f$, FU simplifies to $f-x$ and $AU = f-(f-x) = x$.

Now note that all the angle bisectors of ABCDEF coincide with the perpendicular bisectors of the sides of PQRSTU (eg. consider isosceles ΔBPQ). It therefore follows that the perpendicular bisectors of the sides of PQRSTU are concurrent (at the incentre of ABCDEF) and that it is a cyclic hexagon. As before, we therefore also have for a circum hexagon with a fixed incentre, that the circumcircles arising from different choices of P are concentric with the incircle.

Dual result

Construct any angle divider $\overrightarrow{AP}$ of $\angle A$ of a cyclic hexagon ABCDE, angle divider $\overrightarrow{BP}$ of $\angle B$ so that $\angle PBA = \angle PAB$, angle divider $\overrightarrow{CQ}$ of $\angle C$ so that $\angle QCB = \angle PBC$ and $Q \in \overrightarrow{BP}$, angle divider $\overrightarrow{DR}$ of $\angle D$ so that $\angle RDC = \angle QCD$ and $R \in \overrightarrow{CQ}$, angle divider $\overrightarrow{ES}$ of $\angle E$ so that $\angle SED = \angle RDE$ and $S \in \overrightarrow{DR}$ and angle divider $\overrightarrow{FT}$ of $\angle F$ so that $\angle TFE = \angle SEF$ and $T \in \overrightarrow{ES}$. If U is the intersection of $\overrightarrow{FT}$ and $\overrightarrow{AP}$, then $\angle UFA = \angle UAF$ and PQRSTU is a circum hexagon.

Proof

Consider 4.68b. Let $\angle PAB = x = \angle PBA$, then $\angle QBC = \angle B - x = \angle QCB$, $\angle RCD = \angle C - \angle B + x = \angle RDC$, $\angle SDE = \angle D - \angle C + \angle B - x = \angle SED$, $\angle TEF = \angle E - \angle D + \angle C - \angle B + x = \angle TFE$ and $\angle UFA = \angle F - \angle E + \angle D - \angle C + \angle B - x$. But the two sums of alternate angles are equal for a cyclic hexagon as shown in *Questions and Problems 4*, No. 17, ie. $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$. By substitution of $\angle B + \angle D + \angle F$ by $\angle A + \angle C + \angle E$, $\angle UFA$ simplifies to $\angle A - x = \angle UAF$.

Now note that all the perpendicular bisectors of the sides of ABCDEF coincide with the angle bisectors sides of PQRSTU (eg. consider isosceles ΔQBC). It therefore follows that the angle bisectors of PQRSTU are concurrent (at the circumcentre of ABCDEF) and that it is a circum hexagon. As before, we therefore also have for a cyclic hexagon with a fixed circumcentre, that the incircles arising from different choices of P are concentric with the circumcircle.

Can you now further generalize these two results to circum and cyclic *2n*-gons using the results and notation developed in *Questions and Problems 4*, No. 17?

It should furthermore be observed that for a hexagon to respectively have AU = AP or $\angle UFA = \angle UAF$ with the above constructions, it is not necessary for it to be circumscribed or cyclic, and is it only necessary for the two sums of alternate sides or angles to be equal. For example, this would also be true for the kite and isosceles *2n*-gons discussed in *Questions and Problems 2*, No. 16 and *Questions and Problems 4*, No. 16, since the respective sums of their alternate sides and angles are also equal. However, the obtained *2n*-gon would then not necessarily be cyclic or circumscribed.

(b) Both results are true for any triangle if one goes around the triangle twice (see Figure 4.69).

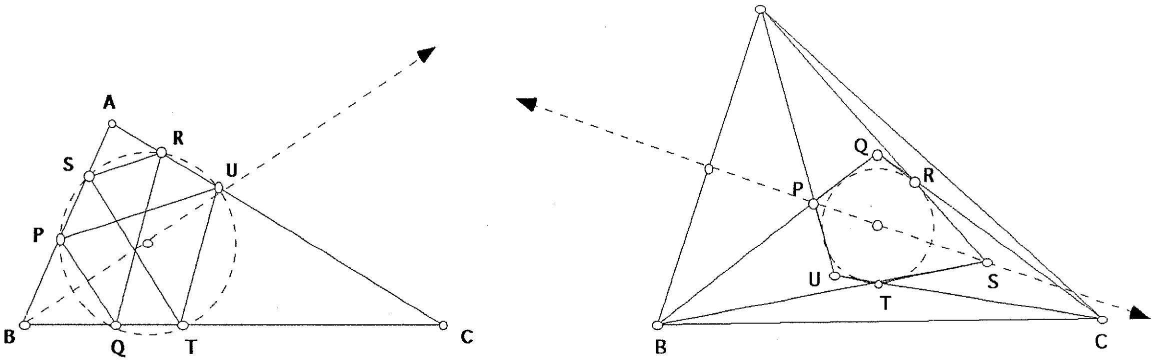


Figure 4.69

First result

Select any point P on AB of a triangle ABC. Take Q on BC so that BQ = PB, R on CA so that CR = QC, S on AB so that AS = RA, T on BC so that BT = SB and U on CA so that CU = TC. Then AU = AP and PQRSTU is a cyclic hexagon.

Dual result

Construct any angle divider $\overrightarrow{AP}$ of $\angle A$ of a triangle ABC, angle divider $\overrightarrow{BP}$ of $\angle B$ so that $\angle PBA = \angle PAB$, angle divider $\overrightarrow{CQ}$ of $\angle C$ so that $\angle QCB = \angle PBC$ and $Q \in \overrightarrow{BP}$, angle divider $\overrightarrow{AR}$ of $\angle A$ so that $\angle RAC = \angle QCA$ and $R \in \overrightarrow{CQ}$, angle divider $\overrightarrow{BS}$ of $\angle B$ so that $\angle SBA = \angle RAB$ and $S \in \overrightarrow{AR}$ and angle divider $\overrightarrow{CT}$ of $\angle C$ so that $\angle TCB = \angle SBC$ and $T \in \overrightarrow{BS}$. If U is the intersection of $\overrightarrow{CT}$ and $\overrightarrow{AP}$, then $\angle UCA = \angle UAC$ and PQRSTU is a circum hexagon.

Can you prove these two results? What else can you say about the hexagon in the first case? (See 25 (b) (cont.) at the end of this chapter).

26. This result can very easily be explained from considering a tetrahedron and folding it flat as shown in Figure 4.70a (compare Kindt, 1993). Reconstructing the tetrahedron by folding up points D, E and F to meet at the top vertex T, it follows that the perpendiculars from D, E and F to the three sides must meet at the foot of the perpendicular from T to the plane ABC.

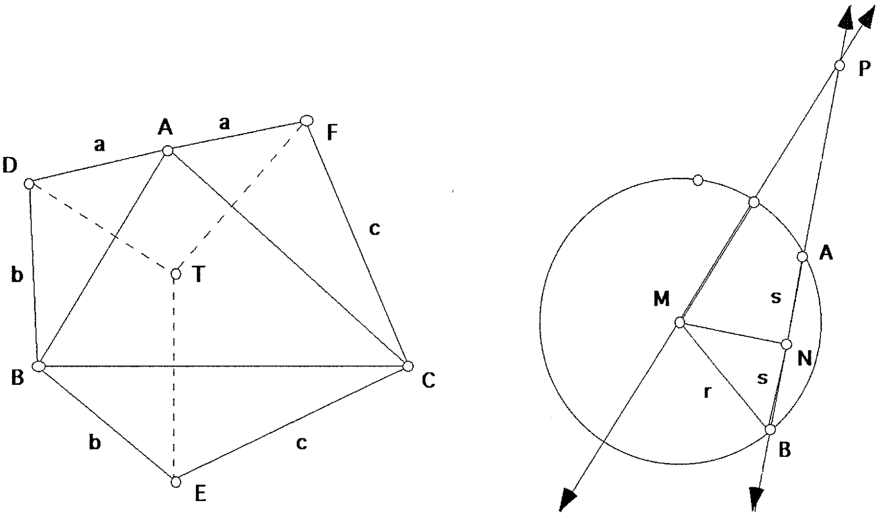


Figure 4.70

However, to prove this result with classical methods we first need the following result.

Lemma

If a line ℓ cuts a circle (M, r) in the points A and B then for every point P of ℓ we have $PA.PB = PM^2 - r^2$.

Proof

Consider Figure 4.70b. MN is perpendicular to chord AB and divides the chord into two equal parts s. Applying Pythagoras twice we have:

$$PM^2 = MN^2 + PN^2$$

$$r^2 = MN^2 + s^2$$

$$\Rightarrow PM^2 - r^2 = PN^2 - s^2 = (PN - s)(PN + s) = PA.PB$$

The number $PM^2 - r^2$ is called the *power* of a point P in relation to the circle (M, r). The power of P is negative, zero or positive depending on whether P lies inside, on or outside the circle. If we have two circles (M_1, r_1) and (M_2, r_2) intersecting in A and B, then for every point B on the line AB the following holds: $PM_1^2 - r_1^2 = PM_2^2 - r_2^2$. In other words, every point P has equal powers in relation to both circles. The converse is also true, namely that every point P with equal powers in respect to both circles must lie on the line AB. That is why line AB is called the *power line* of the two intersecting circles.

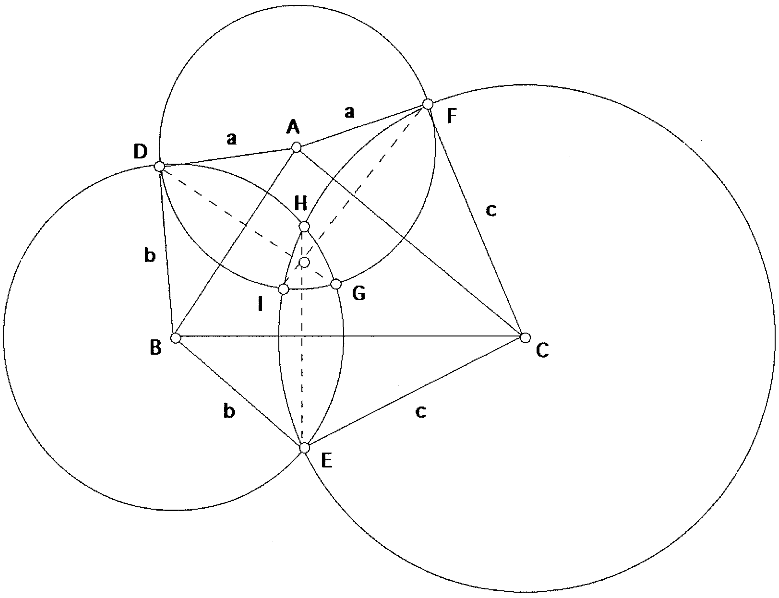


Figure 4.71

Now consider Figure 4.71. Since the three circles each intersect with the other two, we obtain three power lines DG, EH and FI. The intersection point T of say, power lines DG and EH, has equal powers in relation to all three circles and must therefore lie on the third power line FI.

The reader will note that the structure of the above reasoning is similar to loci (symmetry) proofs that the perpendicular or angle bisectors of a triangle are concurrent. (Eg. two perpendicular bisectors meet in a point equidistant from all three vertices and must therefore lie on the third perpendicular bisector. Similarly, two angle bisectors meet in a point equidistant from all three sides and must therefore lie on the third angle bisector).

From the above argument, one can also immediately see that the result would also be valid if the triangles were constructed inwardly. The intersection point of the three power lines is called the *power point* of the three circles.

Can you generalize further to four intersecting spheres constructed on the vertices of a tetrahedron?

24. (continued). The result cannot be generalized to any triangle as can easily be verified by construction and measurement. However, both results can respectively be generalized to a point P on the incircle of a rhombus and a point Q on the circum circle of a rectangle as shown by investigation on *Sketchpad* (see Figures 4.72 & 4.73).

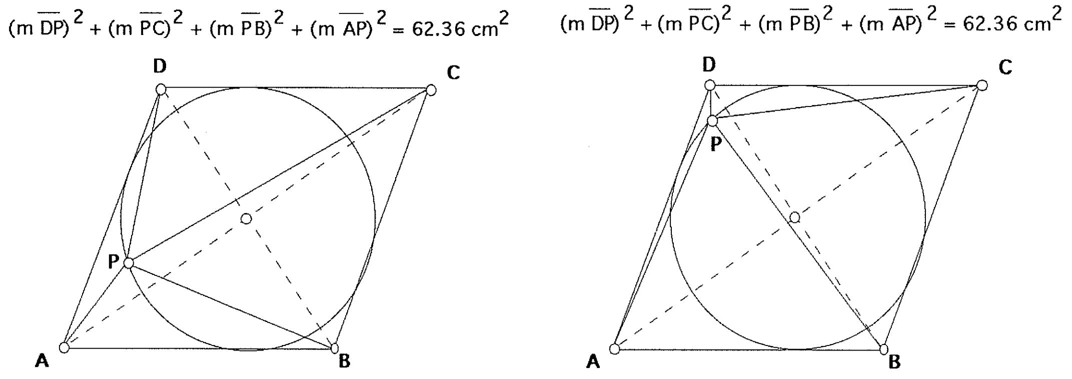


Figure 4.72

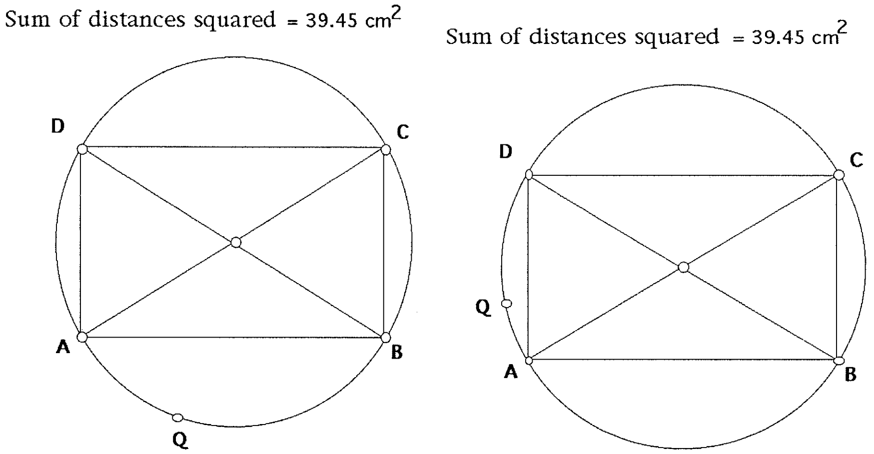


Figure 4.73

Proof of first result

Consider Figure 4.74a where a rhombus with incircle $x^2 + y^2 = a^2$ has been placed with its centre at (0; 0). Let $\angle OCD = \theta$ and P(x; y), then A($-a/\sin\theta$; 0), B(0; $-a/\cos\theta$), C($a/\sin\theta$; 0) and D(0; $a/\sin\theta$) and $PA^2 + PB^2 + PC^2 + PD^2$ $= \left[(x + {}^{a}\!/_{\sin\theta})^2 + y^2\right] + \left[x^2 + (y + {}^{a}\!/_{\cos\theta})^2\right] + \left[(x - {}^{a}\!/_{\sin\theta})^2 + y^2\right] +$

$$\left[x^2 + (y - \tfrac{a}{\cos\theta})^2\right]$$

$$= 4(x^2 + y^2) + 2a^2\left(\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}\right)$$

$$= 4a^2 + \frac{2a^2}{\sin^2\theta\cos^2\theta}$$

$$= 4a^2 + \frac{8a^2}{\sin^2 2\theta}$$

Since a and θ are constant for a fixed rhombus, it follows that the sum of the squares of the distances from P to the vertices is constant. Note that in the case of it being a square, $\theta = 45°$ and the above constant becomes $12a^2$.

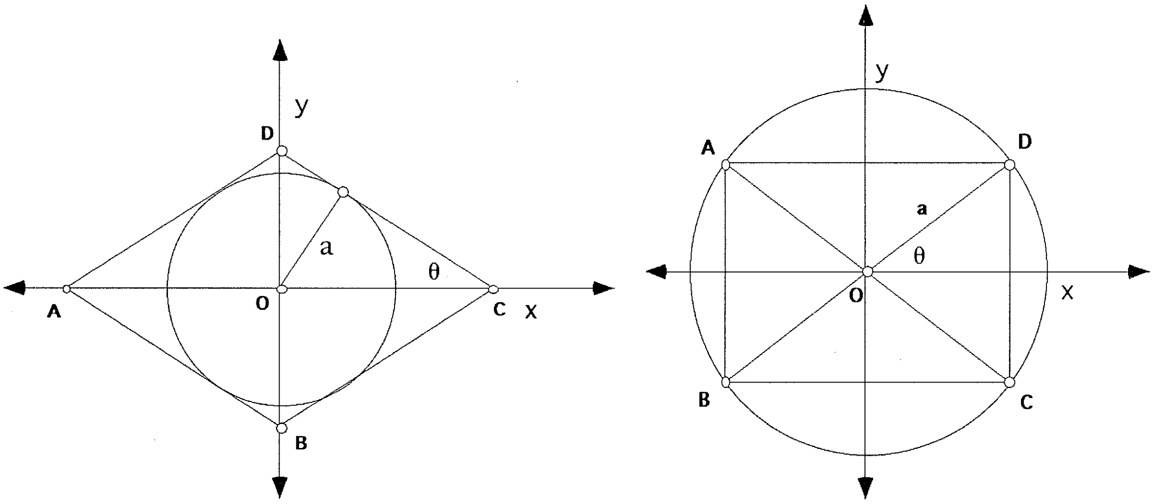


Figure 4.74

Proof of dual

Consider Figure 4.74b where a rectangle with circumcircle $x^2 + y^2 = a^2$ has been placed at (0; 0) as shown. Let Q(x; y) and $\angle DOx = \theta$, then D($a\cos\theta$; $a\sin\theta$) and it follows from symmetry that line AB is $x = -a\cos\theta$, line DC is $x = a\cos\theta$, line AD is $y = a\sin\theta$ and line BC is $y = -a\sin\theta$. Therefore $h_{AB}^2 + h_{DC}^2 + h_{AD}^2 + h_{BC}^2$

$$= (x + a\cos\theta)^2 + (x - a\cos\theta)^2 + (y - a\sin\theta)^2 + (y + a\sin\theta)^2$$

$$= 2(x^2 + y^2) + 2a^2(\sin^2\theta + \cos^2\theta)$$

$$= 4a^2$$

Since a is constant for a fixed circumcircle, it follows that the sum of the squares of the distances from Q to the sides is constant. Interestingly, this constant as shown above, turns out to be independent of θ and therefore of the shape of the rectangle.

Can you generalize these two results further to pentagons, hexagons, etc.?

25. (b) (continued).

Proof of first result

Consider Figure 4.69a. Again let AB, BC and AC be respectively represented by a, b and c. Let PB $= x =$ BQ. Then CQ $= b-x =$ CR, AR $= c-b+x =$ AS, BS $= a-c+b-x =$ BT, CT$= b-(a-c+b-x) = c-a+x =$ CU and AU $= c-(c-a+x) = a-x =$ AP.

Now note that all the angle bisectors of ABC coincide with the perpendicular bisectors of the sides of PQRSTU (eg. consider isosceles ΔBPQ). It therefore follows that the perpendicular bisectors of the sides of PQRSTU are concurrent (at the incentre of ABC) and that it is a cyclic hexagon. As before, we therefore also have for a fixed incentre, that the circumcircles arising from different choices of P are concentric with the incircle of ABC.

Proof of dual result

Consider Figure 4.69b. Let $\angle PAB = x = \angle PBA$, then $\angle QBC = \angle B - x = \angle QCB$, $\angle RCA = \angle C - \angle B + x = \angle RAC$, $\angle SAB = \angle A - \angle C + \angle B - x = \angle SBA$, $\angle TBC = \angle B - \angle A + \angle C - \angle B + x = \angle C - \angle A + x = \angle TCB$ and $\angle UCA = \angle C - \angle C + \angle A - x = \angle A - x = \angle UAC$.

Now note that all the perpendicular bisectors of the sides of ABC coincide with the angle bisectors of PQRSTU (eg. consider isosceles ΔPAB). It therefore follows that the angle bisectors of PQRSTU are concurrent (at the circumcentre of ABC) and that it is a circum hexagon. As before, we therefore also have for a fixed circumcentre, that the incircles arising from different choices of P are concentric with the circumcircle of ABC.

Also note that the hexagon in Figure 4.69a is a parallel-hexagon (opposite sides are parallel - compare Figure 2.31 and Chapter 6). Can you explain why? Can you now further generalize these two results to $(2n-1)$-gons?

Solutions 8

1. It is possible to further generalize the Von Aubel generalization given in Figure 113 to the construction of two pairs of similar *bi-diagonal* quadrilaterals with $\alpha + \beta = 180°$ and arranged on each pair of opposite sides as shown in Figure 8.1. (A bi-diagonal quad is any quadrilateral ABCD with diagonals AC and BD intersecting at O so that two or more adjacent line segments of the four line segments AO, OB, OC and OD are equal. An obvious special case is an isosceles trapezium).

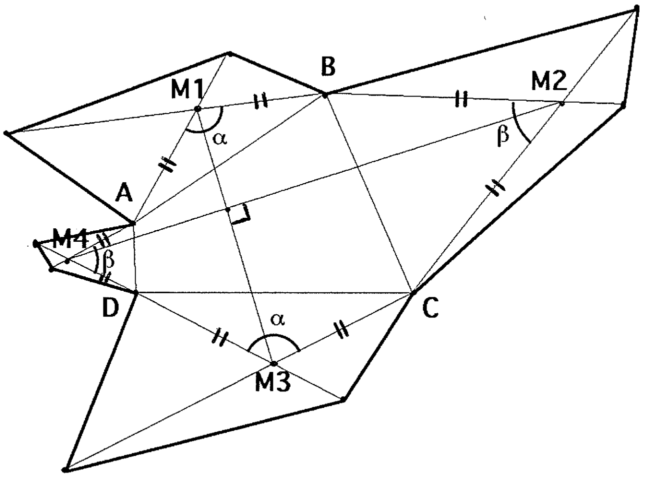


Figure 8.1

Similarly it is possible to further generalize the Von Aubel generalization shown in Figure 114 to similar perpendicular quads arranged as shown in Figure 8.2. As can be seen, the duality between perpendicular and diagonal quads is unfortunately not maintained here. However, the configuration in Figure 8.1 can easily be specialized to two pairs of similar isosceles trapezia on the opposite sides and the one in Figure 8.2 to similar kites.

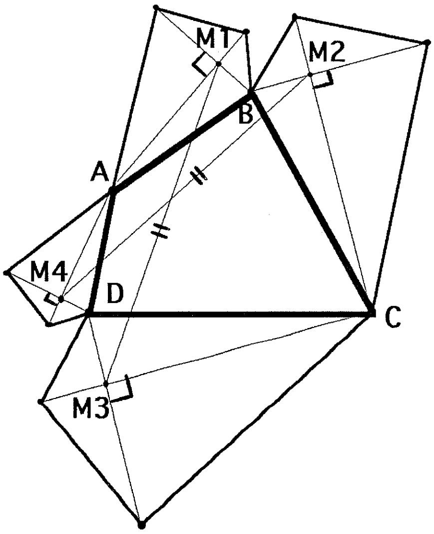


Figure 8.2

The configuration shown in Figure 8.1 can for example also be specialized as shown in Figure 8.3 so that all four bi-diagonal quads are similar to each other. (In this case they are bi-diagonal bisecting quads. Also note that they have at least one right angle. Why?).

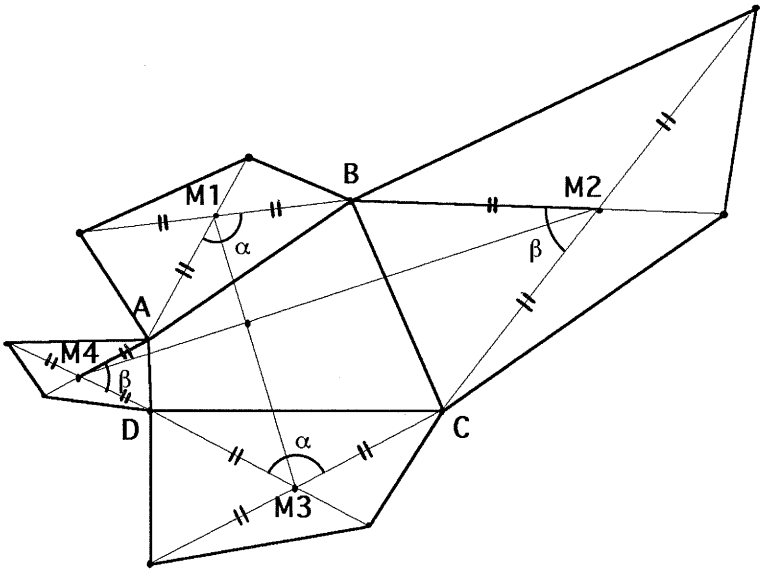


Figure 8.3

2. The author recently mentioned the two Von Aubel generalizations in Figure 113 and 114 to Dirk Laurie, Department of Mathematics, University of Potchefstroom, who proved them very concisely by using vectors as follows.

Given four points A, B, C and D, let $\overline{E}' = \frac{\overline{A}+\overline{B}}{2}$, $\overline{F}' = \frac{\overline{B}+\overline{C}}{2}$, $\overline{G}' = \frac{\overline{C}+\overline{D}}{2}$ and $\overline{H}' = \frac{\overline{D}+\overline{A}}{2}$. The operation to rotate a vector $\overline{X}$ counter-clockwise through an angle s will be indicated by the notation $R[s]\overline{X}$.

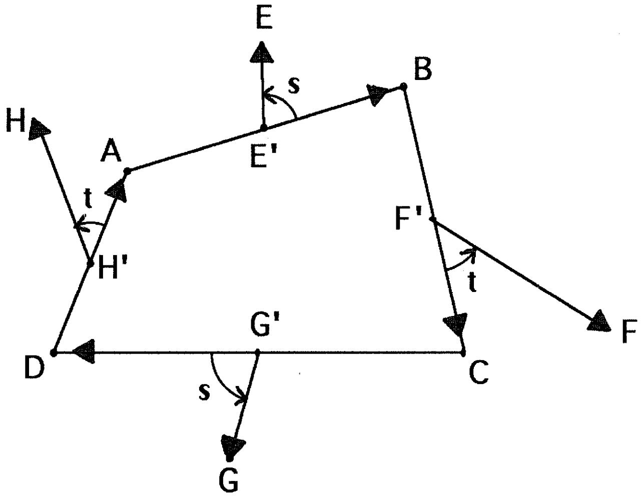


Figure 8.4

Given two angles s and t, and two numbers j and k. Construct point E by rotating vector $\overline{B}-\overline{A}$ counter-clockwise through angle s around point E' and multiplying (enlarging) it by the scalar j (see Figure 8.4). The position vector of point E is therefore $\overline{E}=\overline{E}'+jR[s]\frac{\overline{B}-\overline{A}}{2}$. In a similar fashion construct points F, G and H so that their respective position vectors are $\overline{F}=\overline{F}'+kR[t]\frac{\overline{C}-\overline{B}}{2}$, $\overline{G}=\overline{G}'+jR[s]\frac{\overline{D}-\overline{C}}{2}$ and $\overline{H}=\overline{H}'+kR[t]\frac{\overline{A}-\overline{D}}{2}$. Therefore: $\overline{E}-\overline{G}=\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}+jR[s]\frac{\overline{B}-\overline{A}+\overline{C}-\overline{D}}{2}$ and $\overline{H}-\overline{F}=\frac{\overline{A}+\overline{D}-\overline{B}-\overline{C}}{2}+kR[t]\frac{\overline{A}-\overline{D}-\overline{C}+\overline{B}}{2}$.

Third case (Figure 113)

The outer figures are rectangles, constructed alternately in ratio. Therefore $s=t=90°$ and $k={}^1\!/_j$. Rotate $\overline{E}-\overline{G}$ through $90°$. This gives

$$\begin{aligned}R[90°](\overline{E}-\overline{G}) &= R[90°]\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}+jR[90°+90°]\frac{\overline{B}-\overline{A}+\overline{C}-\overline{D}}{2}\\ &= R[90°]\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}-j\,\frac{\overline{B}-\overline{A}+\overline{C}-\overline{D}}{2}\\ &= j\left\{({}^1\!/_j)R[90°]\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}+\frac{-\overline{B}+\overline{A}-\overline{C}+\overline{D}}{2}\right\}\\ &= j\left\{\overline{H}-\overline{F}\right\}.\end{aligned}$$

Therefore $\overline{E}-\overline{G}$ is perpendicular on $\overline{H}-\overline{F}$.

Fourth case (Figure 114)

The outer figures are rhombi, constructed alternately in direction. Then $j=k=1$ and $t=180°-s$. Rotate $\overline{E}-\overline{G}$ through the angle t. This gives

$$\begin{aligned}R[t](\overline{E}-\overline{G}) &= R[t]\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}+R[t+s]\frac{\overline{B}-\overline{A}+\overline{C}-\overline{D}}{2}\\ &= R[t]\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}+R[180°]\frac{\overline{B}-\overline{A}+\overline{C}-\overline{D}}{2}\\ &= R[t]\frac{\overline{A}+\overline{B}-\overline{C}-\overline{D}}{2}-\frac{\overline{B}-\overline{A}+\overline{C}-\overline{D}}{2}\\ &= \overline{H}-\overline{F}.\end{aligned}$$

Since rotation does not affect the length of a vector, we have that vectors $\overline{E}-\overline{G}$ and $\overline{H}-\overline{F}$ are equal in length.

Summary: quadrilateral definitions

(See classifications and duals on pp. 154-155)

Angle bisecting quad - any quadrilateral with at least one of its angles bisected by a diagonal.

Angle quad - any quadrilateral with at least one pair of equal opposite angles.

Bi-diagonal quad - any quadrilateral ABCD with diagonals AC and BD intersecting at O so that at least two adjacent line segments of the four line segments AO, OB, OC and OD are equal.

Bisecting quad - any quadrilateral with at least one of its diagonals bisected by the other.

Circum side quad - any side quad circumscribed around a circle.

Circum quad - any quadrilateral circumscribed around a circle.

Concave quad - any quadrilateral with one diagonal falling outside the figure.

Convex quad - any quadrilateral with no diagonal falling outside the figure.

Crossed quad - any quadrilateral with both diagonals falling outside the figure.

Cyclic angle quad - any angle quad inscribed in a circle.

Cyclic quad - any quadrilateral inscribed in a circle.

Diagonal quad - any quadrilateral with equal diagonals.

Isosceles circum trapezium - any isosceles trapezium circumscribed around a circle.

Isosceles trapezium - any quadrilateral with at least one axis of symmetry through a pair of opposite sides.

Kite - any quadrilateral with at least one axis of symmetry through a pair of opposite angles.

Parallelogram - any quadrilateral with both pairs of opposite sides parallel.

Perpendicular quad - any quadrilateral with perpendicular diagonals.

Rectangle - any quadrilateral with axes of symmetry through each pair of opposite sides.

Rhombus - any quadrilateral with axes of symmetry through each pair of opposite angles.

Right kite - any kite inscribed in a circle.

Right quad - equivalent to cyclic angle quad (see above).

Side quad - any quadrilateral with at least one pair of equal opposite sides.

Skew bisecting quad - any bisecting quad with at least one pair of equal adjacent angles.

Skew circum quad - any skew isosceles quad circumscribed around a circle.

Skew cyclic quad - any skew kite inscribed in a circle.

Skew isosceles quad - any quadrilateral with at least one pair of equal adjacent angles.

Skew kite - any quadrilateral with at least one pair of equal adjacent sides.

Skew trapezium - any trapezium with at least one pair of equal adjacent sides.

Square - any rhombus with a right angle.

Trapezium - any quadrilateral with at least one pair of opposite sides parallel.

Triangular kite - any kite with at least three equal angles.

Trilateral trapezium - any isosceles trapezium with at least three equal sides.

Glossary

(ABC). Area of ΔABC.

affinity. A transformation which preserves the parallelism of corresponding lines.

altitude of a triangle. Line segment from a vertex perpendicular to an opposite side (or its extension).

angle bisector. Locus of points equidistant from the rays (sides) of an angle.

centroid of a triangle. Point of intersection of its medians.

cevian. A line segment joining a vertex of a triangle to a point on the opposite side (or on its extension).

circumcentre of a cyclic polygon. Centre of its circumscribed circle (point equidistant from its vertices, ie. point of concurrency of the perpendicular bisectors of its sides).

circumcircle of a polygon. Circle circumscribed around a polygon; ie. through its vertices.

circum polygon. A polygon whose sides are tangent to a circle.

collinear. Lie in a straight line.

concurrent. Meet in the same point.

cyclic polygon. A polygon whose vertices lie on a circle.

directly similar. Two figures are directly similar if the similarity transformation which maps the one on to the other preserves angles in both magnitude and direction.

duality. The mathematical principle that all theorems relating to certain concepts have corresponding dual theorems by substituting these concepts by their dual concepts.

enlargement. A transformation where all distances are multiplied by the same scale factor k, ie. is a similarity.

half-turn. A rotation by 180°.

ideal line. The line in the projective plane on which the vanishing points of pairs of parallel lines lie.

incentre of a circum polygon. Centre of its inscribed circle (point equidistant from its sides, ie. point of concurrency of its angle bisectors).

incircle of a polygon. Circle inscribed in a polygon so that its sides are tangent to it.

isometry. A length-preserving transformation; ie. preserves congruency.

isosceles 2n-gon. A 2*n*-gon with an axis of symmetry through one pair of opposite sides.

kite 2n-gon. A 2*n*-gon with an axis of symmetry through one pair of opposite angles.

lemma. A theorem in support of a more major theorem.

orthocenter of a triangle. Point of intersection of altitudes.

perpendicular bisector. Locus of points equidistant from endpoints (vertices) of line segment (side).

prism. A solid with two parallel congruent polygons as opposite faces with edges joining corresponding vertices so that the remaining faces are parallelograms.

projective plane. Euclidean plane plus one ideal line.

rotation. An isometry consisting of turning a figure about a fixed point in the plane.

shear. An affinity (affine transformation) which preserves area.

similarity. A transformation that preserves ratios of distances; ie. shapes of figures.

spiral similarity. The sum of a rotation and an enlargement, or vice versa.

sum (or product) of two transformations. The result of applying the first transformation and then the second.

transformation (of the plane). A mapping of the plane onto itself such that every point P is mapped into a unique image P'.

translation. A transformation such that the directed segments joining points to their images all have the same length and direction.

vanishing point. The ideal point in the projective plane where two parallel lines meet.

2n-gon. A polygon with an even number of vertices, eg. quadrilateral (n=2), hexagon (n=3), etc.

(2n-1)-gon. A polygon with an odd number of vertices, eg. triangle (n=2), pentagon (n=3), etc.

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The purpose of this book is to actively involve the reader in the heuristic processes of conjecturing, discovering, formulating, classifying, defining, refuting, proving, etc. within the context of Euclidean geometry. The book deals with many interesting and beautiful geometric results, which have only been discovered during the past 300 years such as the Euler line, the theorems of Ceva, Napoleon, Morley, Miquel, Varignon, Van Aubel, etc. Some original results are also presented that have not been been published elsewhere before. Extensive attention is given to the classification of the quadrilaterals from the symmetry of a side-angle duality, and covers convex, concave and crossed cases. Several results are generalized to n-gons, 2n-gons and 2n-1-gons respectively.

The book starts off with a brief philosophical and educational overview of the heuristic processes mentioned above. This is intended only as background to the main mathematical explorations, that are contained mainly in the exercises, the solutions of which take up a little more than half the book. The book is not intended for relaxed armchair reading, but for the active involvement of the reader. It does not one-sidedly focus on the neat, tidy aspects of mathematics as a finished product, but tries to illuminate the untidy process aspects of mathematical creativity.

Many examples lend themselves excellently for exploration on computer with dynamic geometry programs such as Sketchpad, although it is not essential to have such a program. The reader should be well acquainted with high school Euclidean geometry and transformation geometry, as well as trigonometry. The book is addressed primarily to university or college lecturers involved in the under-graduate or in-service training of high school mathematics teachers, but may also interest teachers who are looking for enrichment material. as well as gifted high school learners.

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