Maddy's Mathematical Reality

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1. Setting the scene

"But, what *is* minus one?"

A student, Arlene, asked me this question, many years ago. It was, and is, a challenging question. Arlene was one of the mature secretaries in the basic mathematics course that I was employed to teach while I was a mathematics student at Berkeley. I still remember the discomfort of not having a nice reply. I could help these students with the 'problems', but this question was different. Arlene was asking an 'essential' question. Not one about process - investigative or algorithmic - but one about the nature of the mathematical object 'minus one'. I can no longer recall what I said to her, though I do have a clear image of her face and of the sinking feeling in my stomach. I expect that my reply to her was procedural - this is how minus one works - or possibly contextual - stories about beneath sea level or about debts.

Although what I actually said to Arlene has gone, her question served as a stimulus for the sort of reflection on the nature of mathematics which was not part of my mathematics curriculum (back in the mid-seventies). Later, as a teacher in Britain, these issues concerning the 'essential' in mathematics arose again. Sometimes this was prompted by a similar student query. At other times, it was prompted by having to implement a new curriculum, which involved analysing the documentation for what the students were 'really' to learn. In either case, it was a call for a teacher to take a philosophical turn, to apply epistemology and metaphysics to issues arising from the practice of teaching. For what Arlene was, perhaps, asking for was insight into the nature of the mathematical object.

The question of the nature of mathematical objects is a traditional philosophical one. And contemporary philosophy of mathematics has developed refinements of the folkloric 'platonic' position: one such is due to the north American philosopher, Penelope Maddy.

Maddy's theory in brief is that mathematics is part of the physical world - parts of it are perceptible! This is a core claim of her 'set-theoretic realism' which she has been developing since the early 1980s. This direct perception of sets allows the modern foundation of mathematics - set theory - to have a perceptual component. This perceptual core for mathematical foundations serves as a link between the physical world and mathematical abstractions. Links between these two domains are, of course, of interest in mathematics education.

2. Realism

Maddy's philosophy of mathematics is a form of mathematical realism; indeed, her 1990 book is called *Realism in Mathematics* and provides, together with some other papers of hers, the background material for this article. Early in her book, Maddy states:

> Realism, then (at the first approximation), is the view that mathematics is the science of numbers, sets, functions, etc. [... it] is about these things, and the way these things are is what makes mathematical statements true or false. (1990a, p. 2)

It is interesting to have a look at such a theory for, at present, current theories in mathematics education which consider epistemology or ontology of mathematics do not countenance realism. This is evidenced, for instance, by Anna Sierpinska and Stephen Lerman's 1996 review entitled 'Epistemologies of mathematics and of mathematics education'. Their chapter presents a selection of 'epistemologies of mathematics', but does not report on the area I wish to touch on here, i.e., the possibility of a realist ontology underpinning the mathematics we teach.

Their categorisations of epistemologies of mathematics are classified, broadly as:

(a) knowledge through mathematical justification;
(b) knowledge as a function of societies;
(c) knowledge as meaning.

My exegesis of Maddy's work could be seen to contribute to a fourth category:

(d) knowledge of mathematics through science, i.e., from experience in the natural world.

The nearest Sierpinska and Lerman come to mentioning this sort of view is in giving an account of one work by the philosopher Philip Kitcher (p. 836), even though Kitcher's neo-Millian empiricism is not mentioned. The authors use Kitcher's work to exemplify philosophical research into the function of the mathematical community in transmitting mathematical knowledge. And his work offers an excellent example of this! However, Kitcher, like Maddy, is an ontologist as well as an epistemologist; his theory attempts to explain mathematical knowledge and this includes some exposition of what mathematics is and, in Kitcher's words:

> The account [of mathematical knowledge I] offer [may] be viewed as a type of realism. (1984, p. 58)

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Before going any further, the terms 'realism' and its contrary, 'anti-realism', coined by Michael Dummett (e.g., 1992), require further explanation. 'Realism' is a technical philosophical term. Like any term that has a wide usage, it is associated with a rich set of meanings for which no definition could capture its range of nuance. Connotations of the ordinary language notion of 'realistic' are to be eschewed; philosophers, such as Dummett, are fervent anti-realists in this sense. But it is not necessarily un-realistic to be anti-realist! Dummett makes the realist/anti-realist distinction in mathematics to be about sentences and the possibility of ascertaining their truth values. Whether one agrees with that position or not, in the following quotation he pinpoints his distinction between realism ('the way things are') and anti-realism ('the truth about our assertions'):

> It appear[s] to me evident [...] that, interesting as the questions about the nature of mathematical objects, and the ground of their existence, may be, the significant difference lies between those who consider all mathematical statements whose meaning is determinate to possess a definite truth value independently of our capacity to discover it, and those who think that their truth or falsity consists in our ability to recognise it. (1992, p. 465)

Maddy's theory takes truth to be externally determined. Her commitment to this 'realism' comes from two sources. On one hand, following Quine, Maddy considers science: 'our best knowledge of the world' and so aims to use science as an epistemological foundation. On the other hand, she acknowledges the ontological problems ensuing from the 'pre-theoretic realism' (1990a, pp. 1-5) arising from within the phenomenon of mathematical practice. This 'pre-theoretic realism' is well known. It is essentially the phenomenon Anna Sfard (1994) speaks of:

> [The] 'natural' state [of a mathematician's mind is] the state of a Platonic belief in the independent existence of mathematical objects, the nature and properties of which are not a matter of human decision. (p. 51)

Maddy's job, as a philosopher, is to take the data that include reports of such experience and to make a philosophical account. A job for a mathematics educationalist is to scrutinise such a theory for its potential to give insight for the learning of mathematics.

3. Background to Maddy's realism

'Quine/Putnamism'

I have already made reference to Quine as an influence on Maddy; this influence comes chiefly through his theories of naturalised epistemology. Briefly, this is the idea that we cannot stand outside our theories of the world:

> The old epistemology aspired to contain, in a sense, natural science; it would construct it from sense data. Epistemology, in its new setting, conversely, is contained in natural science as a chapter of psychology. [...] There is reciprocal containment [...] epistemology in natural science and natural science in epistemology. (Quine, 1969, p. 83)

Maddy reports that Quine rejects the double standard of considering mathematical entities as purely linguistic, but physical entities as real. And that this rejection is compatible with Putnam's view (e.g. Putnam, 1979) that contemporary physics could not have even been formulated without mathematics. Thus, these philosophers endorse an 'indispensability thesis': mathematics is indispensable for science (e.g. Maddy, 1989, pp. 1132-1133). In other words:

> successful applications justify, in a general way, the practice of mathematics. (Maddy, 1990a, p. 34)

So, if it is granted that advanced scientific theories have an integral mathematical component (quantum mechanics is a standard example), then if the science holds, the mathematics is also confirmed. 'Quine/Putnamism' is the term Maddy coins for their judicious amalgam of 'common-sense' and 'scientific' realism: what common-sensibly exists are those medium-sized objects of our perceptual experience and:

> the considered judgement of science is the best justification we can have. (1990a, p. 13)

This is, thus, a 'two-tier' theory without a particularly mathematical component: the lower tier is common-sense realism, and the higher tier is scientific realism in the form of current science.

Gödel's Platonism

Gödel also advocated a two-tier philosophy of mathematics which was similar to Quine/Putnam's on the higher level. Unlike Quine and Putnam, however, Gödel's 'lower tier' was tied to mathematics. This lower tier consisted of intuitions of mathematical truths. These necessary truths were perceived by a faculty analogous to that of sense perception in science. As Maddy puts it:

> the simpler concepts and axioms are justified intrinsically by their intuitiveness; more theoretical hypotheses are justified extrinsically, by their consequences. This second tier leads to departures from traditional Platonism similar to Quine/Putnam's. Extrinsically justified hypotheses are not certain, and, given that Gödel allows for justification by fruitfulness in physics as well as mathematics, they are not *a priori* either. (1990a, p. 33)

Gödel was significant to Maddy (1990b, p. 266). For, after all, she wanted to explain the ontology involved in mathematical practice and here was a mathematician, with impeccable credentials, who advocated a form of realism. Gödel's philosophy contained the 'flabby' (1990a, p. 35) notion of intuition at its lower tier but concurred with Quine/Putnamism at the higher tier. How can Gödel's theory be made philosophically rigorous? Maddy's 'set-theoretic realism' is an attempt to keep the spirit of Gödel's philosophy within a stricter analytic philosophic framework. She aims to replace the lower tier (of either theory) by set theory perceived. In this way, she could satisfy both Gödel and Quine/Putnam: the lower tier is both mathematics and science.

Realism and epistemology: the problem of access

Without begging the question about the existence or otherwise of mathematical objects, there is a well-received sense in which mathematics is supposed to be about abstractions. Roughly, it means that what one works with in mathematics are not space-time particulars - for example, the motion of a ball through the air may be modelled by a parabola (given requisite parameters). The particular motion of the ball is indeed a space-time particular, but the structure of the model is a mathematical abstraction: parabola. The members of the class of such abstractions are 'mathematical objects'.

The classic question, going back to Plato, is 'how do we get to know about these abstractions if they are not part of our experienced world?' For although the path of the ball is part of the physical world, the parabola-as-model is not. Plato's answer was that we recollect them from a prior existence (see the Meno trans. Guthrie, 1956, pp. 138-9) - hardly an acceptable epistemology in this day and age! Theories of knowledge evolve in time and in the 1970s a theory known as the causal theory of knowledge (see Benacerraf, 1973) was in vogue with analytic philosophers. This theory asserts that:

> (i) there must be a causal connection between that which makes a belief true and the epistemic subject holding the belief.

Now, think back to these 'mathematical objects'. They are abstract. This means that:

> (ii) they are causally inert.

They do not change, like the energy of a moving particle on interaction, nor do they stimulate the senses as my cup of coffee does. Benacerraf noticed that (i) and (ii) implied that there is no possible knowledge of mathematical objects! Although Maddy observes, in her 1984 paper, that the causal theory of knowledge has been refined to a 'reliabilist' theory, this 'Benacerrafian syllogism' captures quite nicely the traditional and pervasive problem of gaining access to the subject matter of 'ideal' subject matter typical of mathematics.

Physicalistic Platonism

In 1990, a collection of articles entitled *Physicalism in Mathematics* was published as a result of a conference, held at the University of Toronto. Maddy's article (1990b) in this collection is entitled 'Physicalistic Platonism'. This title is a provocative oxymoron which indicates her determination to work with truly mathematical entities within science. Maddy chooses to develop her theory under the 'platonistic' label even though her theory does not concur with all of the features of Platonism listed by Irvine in his introduction to the collection (Irvine, 1990, p. xix).

In particular, Maddy rejects (ii) on Irvine's list of points characterising Platonism:

> [mathematical] entities are non-physical, existing outside space and time.

She rejects this notion of an inert abstract entity because of the 'accessibility' issue discussed above. In Irvine's terms, Maddy's theory appears to be that of an 'immanent mathematical' realist (p. xx), for she is a realist who wishes to construe mathematical entities naturalistically (i.e., as part of the world). Irvine categorises Maddy's position as 'physicalistic', but not that of a 'transcendent' (i.e. platonic) realist. In contrast to some versions of Platonism, Maddy's demands neither the necessity nor the *a priori* status usually associated with the platonistic label. In short, Maddy develops a theory that sees mathematics as objective and fallible, involving 'entities' which are not outside our experienced world (as Plato's original Forms were and Irvine's condition (ii) retains). Nevertheless, Maddy's motivation to find a physical germ to mathematics certainly locates her in the 'physicalist' camp. Physicalism is described by the nominalist philosopher of mathematics Hartry Field as:

> the doctrine that chemical facts, biological facts, psychological facts, are all explicable (in principle) in terms of physical facts. The doctrine of physicalism functions as a high-level empirical hypothesis, a hypothesis that no small number of experiments can force us to give up. It functions, in other words, in much the same way as the doctrine of mechanism (that all facts are explicable in terms of mechanical facts) once functioned. [...] Mechanism has been empirically refuted; its heir is physicalism. (cited in Maddy, 1990b, p. 261).

Whether 'physicalistic Platonism' makes conceptual sense or not, this much is clear: Maddy is advocating 'real' mathematics in the real world. I now turn to some of the basic details of her theory:

4. Back to realism à la Maddy: set-theoretic realism

Maddy's aim is to find mathematical entities that are directly perceived and that are foundational with respect to mathematics. This will enable her to replace the 'intuition' of Gödel's platonistic theory with a theory that is both philosophically respectable (it does not rely on 'intuition') and mathematically respectable (its foundations are themselves mathematical).

Her candidates for such entities are sets, the objects of set theory. If she can show that some sets are directly perceived, then she can link these objects of basic perception with the sophisticated and abstract mathematical notions that constitute higher mathematics. It is a clever move: set theory underpins much of modern mathematics. Functions, for example, are defined set-theoretically; this implies that, say, spaces of functions and other concepts dependent on 'function' also have a set-theoretic description. This is already well-established mathematics.

The claim is that we can perceive (at least some) mathematical objects *qua* mathematical entities and not as essentially the abstraction of a scientific (say, physical) notion. For example, a pile of stones is a physical object, on a human scale - rather than one on a chemical or geological scale - and the pile can be considered independent of specific cultures, where it might function as a sculpture or cairn. Maddy can perceive it as a set which, being a set, therefore has a number property. So, it would seem that this set-theoretic and perceptual foundation could satisfy the claim that there can be non-metaphorical mathematical objects and they can be known in an analogous way to some physical objects.

How to perceive a set

If perception of a set is a physical phenomenon, it should have a physical explanation. Maddy relies on the neurophysiological theory of Hebb (1949) to justify her claim that:

> we can and we do perceive sets, and that our ability to do so develops in much the same way as our ability to see physical objects. (1990a, p. 58)

Hebb's particular theory is not itself of importance. What is important for the validity of Maddy's theory, from her naturalised epistemological framework, is that there does exist (or, even, could exist) a scientific explanation for how a set might become an object of perception.

We learn to perceive sets by repeated exposure to them. Eventually, when opening an egg box, we perceive the set of three chicks, or whatever, without a count process. The set with number property three is perceived directly; it does not matter what the presentation of the set of three objects is, we can 'see past' the particular trio to the generality of the triple.

To summarise, Maddy asserts the perceptibility of some sets - and set theory is the foundation of modern mathematics - so that the status of the truth value of mathematical propositions will be shot through with the surety of physical objects. If this is accepted, Gödel's 'lower tier' of intuition may be discarded in favour of a scientifically-based explanation of how these mathematical objects are linked to a physical world.

There are, clearly, all sorts of objections this theory must face. Before bombarding it with these, I want to draw out some points that make it attractive:

> 1. It has been designed to be foundational in both a physical and mathematical sense: to this end, it is simple, imaginative and bold.
> 2. Unless you decree that nothing mathematical can be perceived, it is difficult, given the concept of set, to deny outright the claim that a set has been perceived.
> 3. It provides a theory of transition, from the physical, experienced world to a mathematical world underpinned by set theory.

It is this last point that is particularly interesting to me. For making the transition from, loosely, engaging in physical activity to understanding abstract mathematics is central in learning mathematics. A theory, such as Maddy's, allows ontological unity while making this epistemological transition. [1]

Some objections to set-theoretic realism

Sets and classes and numbers

To have confidence in this theory, the central issue of set perceptibility must be scrutinised, i.e., how can Maddy's reader be convinced that mathematical sets are perceptible? She relies on our naïve perception of 'medium-sized' objects. This granted, she notes, as Frege did, that there is no unique number associated with 'that apple on the desk', so we do not perceive numbers directly. After all, that apple consists of many molecules (which we cannot directly perceive) and it also has several pips, many colours on its skin and one stalk (which at least some of us can perceive). Therefore, material objects are not directly instantiations of numbers. Rather, these material objects of our perception can be perceived in terms of sets and those sets have number properties.

That sets have number properties is clear; what is not so clear is that mathematical sets are perceptible. The word 'set' is in common parlance as well as being a mathematical term. I am not convinced that Maddy does not conflate these two meanings. For example, she explores this set-perceptibility through an example of Steve perceiving a set of eggs (whether there are enough for a recipe). Surely this is an ordinary language use of 'set'? How can Steve decide whether he has perceived the mathematical object? I suggest that Maddy has switched language games from that of the kitchen to that of the theorist and back again - not that the boundary between the domain of these two discourses is anything but fuzzy. This objection can be defeated if Maddy can convince the objector that the mathematical notion of set is not as she puts it: 'a linguistic achievement' (p. 64).

In favour of this a-linguistic notion of set, recent work (Wynn, 1992) on new-borns (3-5 days old) suggests that they can distinguish between sets of small numbers of identical dots. This lends weight to Maddy's assertion that the primitive perceptual understanding related to number is pre-linguistic. But it still does not imply that it is a set that has been perceived.

Peter Milne, in his 1994 review of four different physicalist theses in the philosophy of mathematics, denies the possibility of naïve perception of sets. He claims:

> Thinking in terms of sets is something we learn [...] the mathematical notion of a set [is] a concept that will not arise pre-linguistically. (p. 310)

The reason comes down to the distinction between sets and classes. Milne does allow Maddy's notion of set perceptibility, although without much enthusiasm:

> We see the members of sets (sometimes). We may even see sets, for, as I have said, there seems to be no knock-down argument in favour of their being abstract [and so not perceptible]. (p. 311)

But the more serious objection is his argument that it is not a mathematical set that has been perceived but a classification by properties:

> In Maddy's usage sets are determined extensionally by their members: this is the mathematical notion of collection. Classes, on the other hand, are determined intensionally by properties: this is the logical notion of a collection. (p. 310)

From a very young age we classify things by their type; this, Milne claims, is the pre-linguistic concept, not seeing the set-hood of collections. To exemplify: a child looks in a field in which there are three ponies and many sheep. That child might classify the two kinds of animal from the perception of sheep-type and pony-type. In Milne's terms these are, for the child, collections not mathematical sets, even though that child may later learn about sets and perceive sethood given a similar visual stimulus.

Maddy would argue that, for many subjects, the set with number property three is perceived on looking at those ponies. It is an 'impure set' (1990a, p. 156), but a set nonetheless. As for the sheep, they are too numerous to have an at-a-glance number property, but if they form a set, there is an associated number property. But this implies that we can perceive sets without perceiving their number properties. This veils numbers from direct perception again; properties of these sets can only be inferred. This objection does not worry Maddy - she is only aiming for some perceptibility at the lowest tier. If this is granted, then the mathematical machinery of set theory guarantees the security of other number properties.

On another, related, tack, we can ask 'How many sets are perceived when I look at a lone apple on my desk?' This question raises the concern, which I shall only mention, that there is an abundance of these potentially perceptible sets. Maddy acknowledges Charles Chihara's charge that the set-theoretic realist perceives not only the apple on the desk, but the set consisting of the apple, and the set consisting of the set of the apple, *ad infinitum* (Maddy, 1990a, pp. 150-2; Chihara, 1990, p. 201). Her response is to offer these options: either 'one can't perceive the difference between a singleton and its unit set' or 'deny there is any difference between a set and its singleton' (p. 152). My response to Chihara's objection is that this 'abundance' is a positive advantage to set-theoretic realism. For it permits the infinity of mathematical objects from the finitude of physical ones.

Can the question of whether mathematical sets are perceptible be an empirical question? I think not. I am quite persuaded that infants and animals may be able to make number distinctions without language, but I cannot infer from that-as-a-fact that it is a set which they perceive. Nor can I infer that it is not a set. There are Hebb-type neurological connections in the brain for all sorts of things that we recognise, mathematical or otherwise. This comes down to the nub of realism: are notions like sets 'just' linguistic or are they part of the intrinsic structure of the discreteness humans and other animals can perceive? Maddy's theory develops from the latter; Dummett's from the former.

Triangles, transformations and other mathematical objects

Different branches of mathematics have their own character. By this I mean there are types of question, ways of thinking and modes of justifying that are different across the subdisciplines of mathematics. This is a feature of mathematical practice. In particular, set theory has its own character which is different from that, say, of analysis. Although set theory underpins analysis, the set-theoretic definition of 'function', for example in Halmos (1960, p. 30), gives little clue to how functions function in mathematical practice: the way function theorists think about functions is, in practice, different from the way graph theorists do. Maddy acknowledges this notion of mathematical character:

> Even if the objects of, say, algebra are ultimately sets, set theory does not call attention to their algebraic properties, nor are its methods suitable for approaching algebraic concerns. (1990a, p. 5)

But, thereafter, she does not attend to these aspects of mathematical practice that involve different ways (set-theoretic, algebraic, topological, etc.) of mathematical knowing. Maddy would not consider this divergence of conceptualisation a problem, because, as it is mathematics, each socio-semantic conception (of function) can be traced, in theory, back to a set-theoretic definition. This common core gains its reality from the perception of elementary 'impure sets' and the efficacy of mathematical practice in scientific achievement. I accept this, but Maddy does claim to "develop and defend" (1990a, p. 3) a type of mathematical realism that concurs with the naïve philosophical sense that working mathematicians are supposed to hold. I do not think her theory does do this because of this notion of mathematical character that she recognises but does not develop (this point is explored further in the next section).

Questions about 'converging ontologies'

Maddy, nevertheless, continues to consider mathematical - e.g., geometrical - objects simultaneously as set-theoretic constructions and as meaningful entities within a geometric context. But is not a triangle's reality not just a function of its set-theoretic description, but of its perceptual impact (à la Hebb, for example,) and its functional role (for stability in structures like bridges, say)? Indeed, it is difficult to be sure what would constitute a satisfactory set-theoretic description of a triangle and how this lengthy, precise description would then relate to triangles as used by mathematicians (including learners of mathematics). Maddy's response to the request to give a set-theoretic account of a triangle is likely to be: it can be done in theory. So yet again the practitioner's experiential understanding is in tension with a non-practitioner's theory.

The ontological relationship between sets and mathematical entities is only developed in the case of numbers. Clearly, exemplification of Maddy's theory within every part of mathematics would not be manageable. Nevertheless, the mathematical fields of, for example, geometry, probability and algebra employ notions that have 'entity' status; for example, knowledge of 'random numbers' involves a conceptual base that goes beyond that of any set-theoretic definition. [2] It is not clear how Maddy's set-theoretic 'arithmetisation' of mathematics meshes with these other mathematical concepts which may be 'known' in non-set-theoretic, but quasi-perceptual means.

I believe that Maddy would argue that the truth values of mathematical propositions is the crucial issue. And these propositions can be reduced - albeit laboriously - to their set-theoretic equivalencies. Even if this is acceptable, the more crucial point, for the validity of her account, is to say whether, why and how the geometrical object 'triangle' is the set-theoretic object 'triangle' for the purposes of mathematical practice. Maddy does answer the 'whether' and 'why': 'yes it is the same because set theory is a foundation for all of mathematics' is not only her response but also her rationale. The question of 'how' these ontologies converge is not solved.

5. Pedagogy

There are many serious questions to be asked about the relationship of philosophy of mathematics to teaching.

Mathematician René Thom's (1973) well-known claim captures some of the elusive nature of this relationship:

> In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. (p. 204)

Maddy offers a contemporary philosophy of mathematics. What is the pedagogy which might 'rest' on it? Specifically, in terms of the opening scenario, can Maddy's set-theoretic realism answer Arlene's 'But what *is* minus one?' question?

I am not about to associate pedagogy with philosophy in any way that can be interpreted as suggesting a necessary connection between them. Like Andrew Davis (1992, p. 121), I am clear that my view of mathematics is logically distinct from my classroom decision making and I, like Davis, eschew stereotypes which neatly match personifications of pedagogies to fixed views of the nature of mathematics. Be that as it may, a textbook or other teaching medium can be explicit about the nature of specific mathematical objects, negative numbers, in the case in question. Are the objects that they are explicit about objects of set-theoretic realism? If they are, then Maddy's theory could be a foundation on which an answer to Arlene's question can rest.

Here is a first attempt at an answer to her question: 'minus one' is the additive inverse of the generator of the ring of integers. The generator, 1, is perceptible as the number property of the singleton set. (The rest of the explanation relies on the set-theoretic construction of the ring of integers.) I think that this - clearly set-theoretic realist - definition/explanation is unlikely to be sufficiently related to the properties and function of minus one to satisfy Arlene, so that she has a sense of what the number is.

Communicating the sense, properties and function of mathematical entities is, of course, the job of a mathematics teacher. One of Shulman's (1987) categories for the 'knowledge base for teaching' is what he terms 'pedagogical content knowledge'. This he describes as:

> the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organised, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. (p. 8)

An attraction of set-theoretic realism is that both physical experience and formal structures figure. So what pedagogical 'representation' of 'minus one' could capture both the experiential and formal aspects and be likely to satisfy Arlene's question? Before looking at some possible pedagogical representations, I note that the question of whether an experiential base for learning negative numbers is appropriate at all is disputed by both Fischbein and Freudenthal:

> the chapter of negative numbers has to be treated formally from the beginning. In his [Freudenthal's] view this is the first opportunity offered to a pupil to consider mathematical concepts from a formal deductive viewpoint.
> (Fischbein, 1987, p. 102)

The reason that Fischbein gives for this proposal is that:

> I agree, then, with Freudenthal, who claims that the concept of a negative number contradicted the concept of number itself as it had originally been developed in the history of mathematical reasoning. A negative number is a counter-intuitive concept because it apparently contradicts the notion of existence itself - if existence is considered with its practical meaning. (p. 97)

> the problem of (-a)x(-b) is much harder, [...] *because even very fine mathematical minds could not, for a very long time, completely rid themselves of the impact of implicit intuitive models* [... like] *practical manipulations of concrete magnitudes.* (p. 99)

In other words, these august educators celebrate the liberation from experiential intuition as an opportunity for a new stage in a student's mathematical thinking.

My preference is not to make such a clean break between 'practical mathematics' and 'formal mathematics'. I want to be able to facilitate students' increasingly formal thinking, but my experience tells me that it is best done gradually. Maddy's theory is a philosophy of mathematics which recognises perceptual beginnings but is fundamentally formal and abstract. Although I am not totally convinced that Maddy has tied together these two aspects of her theory at a philosophical level, imperfect theories can still be applied quite satisfactorily. And I think that such a philosophy of mathematics could offer a suitable 'resting place' for a pedagogy which aims to embrace student experience and mathematical precision.

Pedagogical representations

The discussion in this sub-section focuses on two pedagogical representations of negative numbers, and assesses them in terms of their compatibility with Maddy's theory and their potential as didactic devices.

There are two obvious starting points from which a pedagogical representation of negative numbers - one compatible with a set-theoretic realist's ontology - may be developed: either, one starts with perception, or one starts with sets. The former starting point suggests a representation-planning question like: 'How can minus one be perceived?', whereas the latter suggests one like: 'What is minus one in terms of set theory?' I describe, below, two pedagogical representations of small negative integers, one visceral and the other ephemeral in character, which have, respectively, been designed to answer the two representation-planning questions above.

A pedagogical representation for minus one perceived

At face value, to perceive 'small' negative numbers, like -1 or -2, is to be able (non-metaphorically) to perceive a lack. In the film 'Stand and Deliver' (Warner Brothers, 1988), the teacher introduces the notion of negative numbers through the concept-image of a hole: you dig 2 feet down, you fill 2 feet back up until you get to zero, the base level. I interpret this teaching as the teacher drawing to his students' attention the possibility of their perceiving negative two: the 2 feet down-ness can be felt and seen; the lack is perceived as a lack. It is a lack to the extent of the gain that is the adjacent pile of earth 2 feet up (or '2 one cubic foot blocks', to forestall the objection that measurements, i.e. real numbers, were being considered instead of integers). While it could be argued that all the teacher is doing is 'delivering' an image, I would counter that he is calling up an experience he is relying on the students to have had or to be able to imagine because of their memories of bodily actions. To be aware that we can never be certain what others are construing is one thing; to deny that when the student steps into the hole s/he will fall is nonsensical; gravitational pull is part of reality. The 'perceptibility' relies on more than sight-sense - it is kinaesthetic too.

Would Maddy accept this as 'negative two perceived'? As she claims the perceptibility of three eggs, so she would accept that the two one-foot cubes removed from the hole were also perceptible; the student could perceive an 'impure set' with number property two. The question is how to interpret the hole! Is the hole just a two-foot negation or is it 'where those two removed blocks go'? At this juncture, the case for perceptibility of a lack becomes less direct, for, in the latter case, a notion of operation has been introduced. (As it was indeed in 'Stand and Deliver', where the teacher demands: "Fill that hole, go on fill that hole: negative two, plus two is ..." and a formerly unmotivated student becomes engaged and answers "Zero".) Even in the former case, the idea of 'level' is necessary and Maddy does not claim to perceive the null set! Consequently, it is not possible to perceive negative two 'itself' directly without implicitly employing an inverse operation as well. [3] The representation is compatible with Maddy's theory, as it uses the perceptibility of two-ness. It does not give a direct perception of the mathematical entity -2; nearly all mathematical entities are not-perceptible within Maddy's scheme which relies on set-theoretic ontology.

This perception-representation is different in motivational intent from the model-representations which fill (at least, British) school text books. Popular model-representations are freezing temperatures (in degrees Celsius) and depths beneath sea-level. This perceptually-based representation above evokes an embodiment of lack, which has expression as a negative number. The application of negative numbers in modelling suggests that student engagement with the new concept (negative numbers) can be motivated because there is some culturally recognisable application. Curiously, the model of 'debt' is relatively rare in contemporary British school textbooks.

A pedagogical representation for minus one as set-theoretic
The 'modern mathematics' movement of the 1960s was a systematic and sincere attempt to give school mathematics a sound mathematical foundation. The objective of the School Mathematics Project (SMP), founded in 1961, was:

> to devise radically new mathematics courses [...] which would reflect [...] the up-to-date nature and uses of mathematics. (SMP, 1965/1971, p. i)

The SMP designed their materials so that:

> in comparison with traditional texts, these texts pay more attention to an understanding of fundamental concepts. (p. v)

In particular, the notion of a set is seen as 'basic' (p. vi).

Times have changed. In the 1985 SMP 11-16 series, which replaced the original SMP texts written in the 1960s, negative numbers are introduced by the ubiquitous thermometer in the Blue series (which is targeted at average 13-year-old students) and by a decontextualised pattern-spotting in the Yellow series (targeted at moderately capable students of the same age). The structural or formal representations which the 1965 texts offered have been replaced by contextual or inductive representations.

So what did the original SMP texts use as a pedagogical representation of a negative number? In fact, the initial introduction to negative numbers is by means of a vectorial representation, (1965/1971, pp. 192-208). The authors assume that the number line representation for counting numbers is understood, then they represent directed numbers as shifts - forwards or backwards - along the number line (independent of initial position, i.e. as a free vector):

> corresponding to every counting number we associate a negative shift number and a positive shift number. (p. 194)

This is a pedagogic representation of the set-theoretic definition of negative numbers, an exposition of which can be found in van der Waerden (1970, pp. 6-7). Indeed, van der Waerden's formal integers hardly differ from the 1965 SMP account. By the mid-1980s, the SMP emphasis is on procedure: both Yellow and Blue books talk about "going up and down" a number line or thermometer, respectively. In both cases, the notation "-1, -2, etc." is 'defined' ostensively.

This vectorial approach has similarities with that of the 'Stand and Deliver' teacher as both evoke the notion of negative number being an inverse operation: "fill that hole!"; "backwards shift!". The difference is in whether the emphasis is on recognising bodily experience or on preparation for formal structure. My assessment of these representations is that there is no hard and fast line between them. The 'visceral' representation embodies an operation and its inverse on the natural numbers. This representation also employs the notion that natural numbers are properties of perceptible sets.

The 'ephemeral' representation uses the set-theoretic structure used to define the integers, which includes the pair of mutually inverse operations + and -. Although, here, the natural numbers' axiomatic foundation is ontologically prior, in practice children will begin their study of number with perceptions of and operations on discrete objects. Maddy's theory can, therefore, be mapped onto either representation. In other words, we do have a possibility for a perceptual beginning for mathematical thinking of a set-theoretically-based type.

6. Concluding remarks

A realist mathematical philosophy implies a non-metaphorical use of 'object' at some level, notwithstanding this word's evident material root. Set-theoretic realism is a theory that explains explicitly in what sense a mathematical object is non-metaphorical. This is not to deny that within the process of development of new mathematical concepts, mathematicians use terms and concepts which are often named metaphorically. A recent mathematics seminar at The

Open University was entitled "Where is the orbit of a discrete group thick?" [4], every significant word of which could be considered to be metaphorical. Sfard's (1994) report of interviews with three Israeli mathematicians illustrates the psychology of such invention. But this use of metaphor neither denies the objecthood of these mathematicians' inventions nor confirms it. It just says that mathematicians like to name aspects of the content of their study with names that are easy to remember - a social phenomenon. Chaos is not a mess but a theory; it might just as well have been called 'butterfly theory'.

Even if the perceptibility of sets is rejected, Maddy's theory gives a flavour of a contemporary realist philosophy of mathematics. In particular, it is a physicalist theory; one that starts from the assumption that mathematics is integral to the physical world, of which human culture forms but a tiny part.

Notes

[1] I paraphrase Steven Rose's remark: "I believe we do live in a world which is ontologically unitary, but to understand it we need epistemological diversity". ('When making things simple does not give the right explanation', *The Times Higher Educational Supplement*, 5th September, 1997, pp. 16-17)

[2] Milne (1994) discusses further the particular problems inherent in set-theoretic realism as it pertains to probability theory.

[3] Kitcher's philosophy of mathematics (Kitcher, 1984) would support pedagogical representations of inverses more easily, for he bases mathematical knowledge on primitive operations on material objects. But that is another story.

[4] By Dr. Torbyn Lundh, a visiting fellow at Cambridge University.

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