The Parabola

Math 1220 (Spring 2003)

To define the parabola (or any conic) in the plane, we choose:

(a) A point F in the plane, called the "focus" and

(b) A line L in the plane, called the "directrix."

Then the parabola with focus F and directrix L is set of points P so that:

$$|PF| = |PL|$$

(i.e. the distance from P to F is the same as the distance from P to L)

It simplifies the algebra considerably if we rotate (and translate) the plane to move the focus and directrix into "standard" position, so that:

$$F = (p, 0) \text{ and } L = \{x = -p\}$$

(that is the line L is vertical, consisting of all points of the form $(-p, y)$).

Then we calculate away:

$$|FP| = \sqrt{(x-p)^2 + y^2}$$
$$|FL| = x + p$$

so setting the two to be the same and squaring gives:

$$(x-p)^2 + y^2 = (x+p)^2$$
$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$$
$$y^2 = 4px$$

which is the **standard equation for the parabola.**

There is an important **optical property** of the parabola, which is:

Optical Property: If $P = (x_0, y_0)$ is a point on the parabola, then the angle between the horizontal line an the tangent line at P is the same as the angle between the line $\overline{PF}$ and the tangent line at P.

See the book for a picture! To prove this, we need the equation for the tangent line. First, we find the slope by implicit differentiation:

$$y^2 = 4px; 2yy' = 4p; y' = \frac{4p}{2y} = \frac{2p}{y}$$

so the slope of the tangent line at $P = (x_0, y_0)$ is:

$$m = \frac{2p}{y_0}$$

Using the fact that the tangent line passes through P, we get the equation:

$$(y - y_0) = \frac{2p}{y_0}(x - x_0)$$

Now let B be the intersection point of the tangent line with the x-axis:

$$B = (b, 0) \text{ where } -y_0 = \frac{2p}{y_0}(b - x_0)$$

so we can solve for b:

$$\frac{-y_0^2}{2p} + x_0 = b$$

but $y_0^2 = 4px_0$ since P is on the parabola. This gives:

$$-2x_0 + x_0 = b$$

so that $b = -x_0$. If we consider the triangle ΔBFP, we see that:

$$|BF| = p + x_0 \text{ and } |FP| = |LP| = x_0 + p$$

so ΔBFP is isosceles, and then the optical property follows.

Physical Interpretation of the Optical Property: If a light bulb is placed at the focus of a parabola with mirror walls, then all the light will reflect horizontally off the walls of the parabola. That is, a parabola makes a great flashlight!