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"Irreversibility, Feasibility Contraction, and
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Etsusaku Shimada

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KYOTO UNIVERSITY

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Etsusaku Shimada

Abstract

Dynamic welfare theory typically relies on two background premises: losses are compensable, and continuation values can be summarized by a state variable sufficient for recursive evaluation. This paper identifies a structural boundary of both premises. When irreversible time loss strictly contracts feasible continuation sets, no finite transfer can restore welfare equivalence, and no state variable that omits feasibility structure remains preference-sufficient. Under opportunity monotonicity and transfer neutrality, strict opportunity contraction therefore yields a joint failure of compensability and state-sufficient representation. The opportunity correspondence emerges as the minimal welfare-sufficient state. The paper further shows that strict opportunity contraction arises generically in dynamically productive environments and that the same mechanism extends to Bellman recursion, social evaluation, and regulated market settings. The contribution is diagnostic rather than revisionist: it isolates a feasibility-based limit of dynamic welfare representation that is logically prior to discounting, aggregation, or curvature assumptions.

JEL Classification: C62, D60, D71, D90

Keywords: Dynamic Welfare Representation, Irreversibility, Opportunity Contraction, Feasibility Correspondence, Compensability, Recursive Representation

1. Introduction

Modern dynamic welfare theory rests on a structural premise. Losses are presumed compensable, and continuation values are presumed representable by a state variable sufficient for recursive evaluation. These two assumptions—compensability and state sufficiency—underlie compensation tests in welfare economics, recursive utility models in intertemporal theory, and the Bellman representation central to dynamic programming (Bellman 1957; Stokey and Lucas 1989; Koopmans 1960). The compensatory reasoning underlying modern welfare economics traces back to the Hicks–Kaldor tradition, in which losses are presumed offsettable by transfers (Hicks 1939; Kaldor 1939), and to ordinal representation results establishing the conditions under which preference orderings admit numerical representation (Samuelson 1937; Debreu 1954). In this respect, dynamic welfare theory inherits two distinct but intertwined strands of analysis: the compensatory tradition of welfare economics and the recursive representation tradition of dynamic choice theory. The former emphasizes transfer-based equivalence; the latter emphasizes informational compression through state variables. The present analysis situates itself at the intersection of these two traditions.

This paper identifies a structural boundary of that premise. When irreversible time loss strictly contracts feasible continuation sets, both compensability and state-sufficient representation fail simultaneously. The point is nonparametric. It does not depend on additivity, continuity, concavity, differentiability, stationarity, or discounting. Rather, it follows from two ordinal conditions: opportunity monotonicity and transfer neutrality. Under these assumptions, strict opportunity contraction implies both the impossibility of finite compensatory transfers and the necessity of the opportunity correspondence as the minimal welfare-sufficient state.

The argument proceeds from a simple observation. Dynamic welfare comparisons depend not only on realized payoffs, but also on which future actions remain feasible. If, at some date, irreversible time loss eliminates a subset of feasible continuations, no finite transfer can reconstruct the lost continuation set. Compensation fails, not because transfers are too small, but because feasibility has changed. Once feasibility contracts strictly, any state variable that omits the opportunity correspondence collapses distinct welfare-relevant structures into the same informational object. In this respect, the paper reverses the usual direction of representation analysis: rather than deriving existence from structural axioms, it identifies a feasibility configuration under which representation must fail.

The central result—Theorem 1—establishes a dynamic welfare representation impossibility. Under opportunity monotonicity and transfer neutrality, strict opportunity contraction implies both the failure of finite compensability and the failure of state-sufficient recursive representation. The opportunity correspondence itself therefore emerges as the minimal welfare-sufficient state.

This boundary is not a knife-edge phenomenon. In standard dynamic environments with productive actions, strict opportunity contraction arises generically. The set of states exhibiting contraction contains a nonempty open subset and is residual within that subset, following the genericity tradition in economic theory (Debreu 1970; Mas-Colell 1974). The resulting representation failure is therefore structurally stable under perturbations.

The analysis extends along three dimensions. First, the failure of state sufficiency implies a breakdown of Bellman recursion whenever observed state variables do not encode feasibility structures. Recursive dynamic programming relies on informational compression of histories into sufficient states. The Markov sufficiency presupposed in recursive models (Bellman 1957; Stokey and Lucas 1989) requires that continuation problems coincide conditional on the state; strict opportunity contraction violates this presupposition at the level of feasibility geometry. When feasibility depends on irreversible time loss not captured by the state variable, Bellman representation becomes inconsistent with opportunity monotonicity.

Second, the boundary persists under social aggregation. If at least one individual experiences strict opportunity contraction without offsetting expansion for others, Pareto opportunity monotonicity implies strict social preference. Yet no finite transfer restores equivalence. The product opportunity correspondence thus constitutes the minimal welfare-sufficient social state.

Third, the structural boundary admits equilibrium realization. In regulated markets with binding price constraints, firms may adjust quality rather than price. When quality reductions shift time burdens onto consumers and contract feasible continuation sets, equilibrium implements opportunity contraction. Dynamic welfare representation impossibility thereby arises within standard market environments.

It is important to clarify what this paper does not claim. The results do not invalidate recursive or additive welfare frameworks within domains where feasibility is stable. Nor do they propose a domain restriction restoring coherence. Rather, the contribution is diagnostic. It identifies the precise feasibility condition under which compensatory reasoning and recursive representation cease to be jointly sustainable. While irreversibility has been extensively studied in environmental and investment theory (Arrow and Fisher 1974; Henry 1974; Dixit and Pindyck 1994), its implication for the possibility of dynamic welfare representation has not been isolated at a purely structural level. Whereas the environmental and investment literatures emphasize option value and timing under uncertainty (Arrow and Fisher 1974; Henry 1974; Dixit and Pindyck 1994), the present analysis isolates irreversibility as a feasibility-based constraint on representability itself.

Irreversibility therefore marks a boundary of dynamic welfare representation. Before questions of aggregation, discounting, or curvature arise, there exists a prior feasibility constraint: the preservation of feasible continuation sets. When that constraint is violated through strict opportunity contraction, no parametric adjustment restores representability. The limitation is structural and precedes domain-based recovery results.

The remainder of the paper formalizes this boundary. Section 2 clarifies the logical architecture linking opportunity monotonicity, transfer neutrality, and representation failure. Section 3 introduces the opportunity-state framework. Section 4 establishes the dynamic welfare representation impossibility theorem and characterizes minimal sufficiency. Section 5 proves the genericity of strict opportunity contraction. Sections 6–8 extend the analysis to Bellman recursion, social aggregation, and market implementation. Section 9 concludes. The purpose of this discussion paper is to clarify the structural position of the representation boundary across recursive, social, and market domains, thereby making explicit the feasibility-based constraint underlying dynamic welfare evaluation.

2. Logical Architecture of Representation Failure

The results of this paper rest on a precise logical structure linking two ordinal premises to two distinct but necessarily connected forms of representation failure. This section clarifies that structure before formalization.

The architecture parallels axiomatic approaches in welfare and social choice theory, where informational requirements are derived from minimal ordinal conditions (Debreu 1954; Maskin 1978). However, the direction of analysis differs fundamentally. Rather than identifying conditions under which representation exists, we isolate a structural configuration of feasibility under which representation must fail.

2.1 Premises

We impose two assumptions on preferences over histories.

First, Opportunity Monotonicity.

If, at some date, one history admits a strict expansion of feasible continuation paths relative to another, then it is strictly preferred. Welfare is strictly increasing in feasible continuation opportunities.

Second, Transfer Neutrality.

Finite monetary transfers do not expand feasible continuation sets. Transfers may alter resource levels but leave the underlying feasibility correspondence unchanged.

Both conditions are purely ordinal. They impose no continuity, convexity, separability, additivity, or discounting structure. Opportunity monotonicity concerns the ordering of feasible sets; transfer neutrality concerns the invariance of feasibility under redistribution.

Taken separately, each assumption is weak. Together, however, they create a structural tension once strict opportunity contraction is admitted.

The key observation is that opportunity monotonicity renders feasibility sets welfare-relevant, while transfer neutrality blocks the compensatory mechanism through which welfare equivalence would ordinarily be restored. Once feasibility contraction occurs, the two premises jointly preclude both redistribution-based recovery and informational compression.

This joint implication does not depend on interpersonal comparability, cardinal utility, or curvature conditions. It arises at the level of feasible correspondences themselves.

2.2 Two Failure Objects

From these premises, two failure objects follow once strict opportunity contraction occurs.

(i) Failure of Finite Compensability

Suppose that, at some date, a history h' admits a strict contraction of feasible continuation paths relative to h. By opportunity monotonicity, h is strictly preferred to h'.

Transfer neutrality implies that any finite transfer applied to h' leaves the contracted feasible set unchanged. The strict inclusion persists. Therefore, the welfare ranking cannot be reversed by any finite transfer.

The failure of compensation is thus structural. It does not arise from insufficient transfers or nonlinear utility curvature, but from irreversible contraction of feasible continuation sets.

More abstractly, compensatory reasoning presumes reversibility of opportunity structure under redistribution. Strict contraction violates that presupposition. The impossibility thus arises prior to any parametric welfare comparison; it arises from the geometry of set inclusion itself.

(ii) Failure of State-Sufficient Representation

Dynamic welfare representation requires that continuation values be summarized by a state variable sufficient for ordering future paths. If two histories share the same state, recursive representation assigns them identical continuation values.

Under strict opportunity contraction, however, two histories may coincide in observable state variables while differing in their feasible continuation sets. If the state variable omits the opportunity correspondence, it collapses distinct welfare-relevant structures into the same informational state. Opportunity monotonicity then implies a strict preference ordering that the representation cannot reproduce. State sufficiency fails.

The two failures are not independent phenomena. They arise from a common structural source: irreversible contraction of feasible continuation correspondences. Compensability fails because redistribution cannot recreate eliminated sets. State sufficiency fails because informational compression cannot ignore eliminated sets. Both failures therefore reflect the same feasibility asymmetry expressed in different evaluative dimensions.

2.3 Minimality of the Opportunity Correspondence

Once strict contraction is admitted, the opportunity correspondence itself becomes the minimal welfare-relevant informational object.

Any state variable that is strictly coarser than the opportunity correspondence—meaning that it assigns identical states to histories inducing distinct feasible continuation sets—necessarily collapses welfare-relevant distinctions.

Thus the opportunity correspondence constitutes the minimal welfare-sufficient state.

This minimality result parallels classical sufficiency results in representation theory, where informational compression is admissible only if it preserves preference orderings (Debreu 1954). Here, however, minimal sufficiency is determined not by separability or continuity conditions but by feasibility structure itself. Informational reduction beyond the opportunity correspondence is incompatible with opportunity monotonicity once strict contraction occurs.

2.4 Structural Boundary

The logical structure may be summarized schematically:

1. Opportunity monotonicity renders feasible continuation sets welfare-relevant.
2. Transfer neutrality blocks compensatory expansion of feasibility.
3. Strict contraction creates irreversible inclusion asymmetry.
4. Redistribution cannot restore eliminated feasibility.
5. Informational compression cannot ignore eliminated feasibility.
6. Compensability and state sufficiency therefore fail jointly.

The boundary identified in this paper is therefore prior to aggregation, discounting, or curvature. It is a feasibility-based constraint on representability itself.

The subsequent sections formalize this architecture. The opportunity-state framework specifies feasible continuation correspondences. Theorem 1 establishes the joint failure of compensability and state sufficiency. Section 5 shows that strict opportunity contraction arises generically. The remaining sections trace the recursive, social, and market implications of this structural boundary.

3. Opportunity-State Framework

This section formalizes the opportunity-state structure underlying the representation boundary.

3.1 Histories and Feasible Continuations

Time is discrete, indexed by $t = 0,1, \dots, T$, where T may be finite or countably infinite.

Let $\mathcal{H}$ denote the set of feasible complete histories.

A history $h \in \mathcal{H}$ specifies a sequence of actions, states, or outcomes over time.

For any partial history h_t realized up to date t, define the opportunity correspondence

$$F_t(h_t) \subseteq \mathcal{H}$$

as the set of feasible continuation histories from date t onward, conditional on h_t.

This formulation parallels state-space constructions in dynamic preference theory, where continuation possibilities are explicitly modeled as sets of future contingencies (Kreps 1979; Dekel, Lipman, and Rustichini 2001). Here, however, the emphasis is not on subjective state spaces but on objective feasibility correspondences.

We write

$$x_t(h) := F_t(h_t)$$

for the feasible continuation set induced by history h at date t.

Thus a complete history can be viewed as generating a sequence

$$h \mapsto (x_0(h), x_1(h), \dots, x_T(h)),$$

where each $x_t(h)$ represents the continuation opportunities remaining at date t.

No convexity, continuity, or measurability assumptions are imposed on the correspondence. The analysis relies only on set inclusion relations.

3.2 Preferences over Histories

Let $\succeq$ denote a complete and transitive preference relation over $\mathcal{H}$.

We impose two ordinal assumptions.

The reliance on completeness and transitivity aligns the analysis with the classical ordinal representation framework (Debreu 1954), while deliberately avoiding cardinal or topological assumptions.

A1. Opportunity Monotonicity

For any histories $h, h' \in \mathcal{H}$, if there exists a date t such that

$$x_t(h) \supset x_t(h'),$$

then

$$h \succ h'.$$

Strict expansion of feasible continuation paths strictly increases welfare.

This assumption captures the idea that larger feasible opportunity sets are strictly welfare-improving, holding all else fixed. The monotonicity condition echoes preference-for-flexibility arguments in dynamic choice theory (Kreps 1979), but here the ordering is induced purely by set inclusion rather than by subjective uncertainty. It does not require utility representation or cardinal structure. The ordering is induced purely by inclusion of feasible continuations.

A2. Transfer Neutrality

For any finite monetary transfer Δ, and any history h,

$$x_t(h + \Delta) = x_t(h) \text{for all } t.$$

Transfers may alter resource levels or contemporaneous payoffs but do not expand feasible continuation sets.

Transfer neutrality ensures that compensation operates only through resource redistribution and does not modify feasibility structure.

In this respect, the assumption isolates the feasibility dimension from the redistributional logic underlying Hicks–Kaldor compensation (Hicks 1939; Kaldor 1939).

3.3 Strict Opportunity Contraction

We now formalize the feasibility condition at the center of the analysis.

Definition 1 (Strict Opportunity Contraction)

Strict opportunity contraction occurs at date $t$if there exist histories $h, h' \in \mathcal{H}$ such that

$$x_t(h') \subset x_t(h).$$

Intuitively, h' involves an irreversible loss—such as time loss or quality degradation—that eliminates feasible continuation paths available under h.

The contraction is strict when inclusion is proper.

3.4 Irreversibility and Time Loss

Irreversible time loss provides a canonical environment in which strict opportunity contraction arises. While irreversibility has been analyzed primarily in the context of investment timing and environmental preservation (Arrow and Fisher 1974; Henry 1974; Dixit and Pindyck 1994), the present formulation abstracts from optimal stopping considerations and focuses on the induced contraction of feasible continuation correspondences.

Time, once lost, cannot be recreated or transferred across dates. If time loss at some date prevents the execution of actions that would otherwise generate future options, then feasible continuation sets shrink. Importantly, strict contraction concerns feasibility rather than realized outcomes. Even if contemporaneous payoffs can be adjusted through transfers, eliminated future actions remain infeasible. Thus irreversibility manifests as contraction of feasible continuation correspondences.

3.5 State Variables and Informational Compression

Dynamic representation typically assumes that welfare over histories can be summarized by a state variable $s_t$such that continuation preferences depend only on s_t. This assumption underlies recursive utility and dynamic programming frameworks (Koopmans 1960; Bellman 1957; Stokey and Lucas 1989), where histories are compressed into payoff-relevant states.

Formally, a state variable is a mapping

$$s_t\colon \mathcal{H} \to \mathcal{S}$$

where $\mathcal{S}$ is a state space.

A state variable is Markov sufficient if, whenever

$$s_t(h) = s_t(h'),$$

preferences over all feasible continuations from date $t$coincide.

The central question of this paper is whether such compression is possible when strict opportunity contraction occurs.

3.6 Summary

The opportunity-state framework isolates three objects:

1. Feasible continuation correspondences F_t.
2. Ordinal preferences satisfying opportunity monotonicity.
3. State variables that attempt to summarize continuation structure.

The subsequent section shows that, under strict opportunity contraction and transfer neutrality, no state variable that fails to encode the opportunity correspondence can preserve welfare ordering.

4. The Structural Representation Boundary

This section establishes the central result of the paper.

Under opportunity monotonicity and transfer neutrality, strict opportunity contraction generates a joint failure of compensability and state-sufficient representation.

Whereas classical representation results identify conditions under which dynamic preferences admit recursive or additive structure (Koopmans 1960; Kreps 1979; Dekel, Lipman, and Rustichini 2001), the theorem below identifies a structural configuration under which such representation is impossible.

We begin by clarifying the relevant notion of sufficiency.

4.1 Concepts of Sufficiency

Dynamic welfare representation typically relies on informational compression of histories into state variables. We distinguish three related notions.

Definition 2 (Informational Sufficiency)

A state variable $s_t: \mathcal{H} \to \mathcal{S}$ is informationally sufficient at date t if, whenever

$$s_t(h) = s_t(h'),$$

the feasible continuation sets coincide:

$$x_t(h) = x_t(h').$$

Informational sufficiency requires that the state encode the full feasibility structure.

This notion parallels informational sufficiency in recursive dynamic models, where continuation problems must be identical conditional on the state (Bellman 1957; Stokey and Lucas 1989).

Definition 3 (Preference Sufficiency)

A state variable is preference-sufficient at date t if, whenever

$$s_t(h) = s_t(h'),$$

preferences over all feasible continuations from date t coincide.

Preference sufficiency is weaker than informational sufficiency. It does not require identical feasible sets, only identical induced preference orderings over continuations.

The distinction mirrors the gap between structural and reduced-form representations in axiomatic theory (Debreu 1954), but here the divergence is driven by feasibility contraction rather than utility representation.

Definition 4 (Minimal Welfare-Sufficient State)

A state variable s_t is minimally welfare-sufficient if:

1. It is preference-sufficient; and
2. Any strictly coarser state variable fails to be preference-sufficient.

Minimality concerns informational compression. A minimally sufficient state contains no redundant information relative to welfare ordering.

4.2 Theorem 1: Dynamic Welfare Representation Impossibility

We now state the central result.

Theorem 1 (Structural Representation Boundary)

Suppose preferences over histories satisfy Opportunity Monotonicity (A1) and Transfer Neutrality (A2). If strict opportunity contraction occurs at some date t, then:

(i) No finite transfer restores welfare equivalence; and

(ii) No state variable that fails to encode the opportunity correspondence F_t is preference-sufficient at date t.

Consequently, the opportunity correspondence constitutes the minimal welfare-sufficient state.

The result complements the classical representation literature by establishing a boundary condition: while Koopmans (1960) and related work characterize when recursive evaluation exists, Theorem 1 characterizes when it cannot exist under ordinal monotonicity and feasibility invariance.

The theorem identifies a structural boundary. Once strict contraction occurs, compensatory reasoning and state compression fail jointly.

4.3 Proof Structure

The proof proceeds in three steps.

Lemma 1 (Persistence of Contraction under Transfers)

If strict opportunity contraction holds between histories $h$and h' at date t, then for any finite transfer Δ,

$$x_t(h' + \Delta) = x_t(h') \subset x_t(h).$$

Proof.

Transfer neutrality implies feasibility sets are invariant under finite transfers. Strict inclusion therefore persists. ■

Lemma 2 (Failure of Finite Compensability)

Under strict opportunity contraction, no finite transfer restores welfare equivalence.

Proof.

By opportunity monotonicity,

$$h \succ h'.$$

By Lemma 1, strict inclusion persists under any finite transfer applied to h'. Opportunity monotonicity therefore implies

$$h \succ h' + \Delta$$

for all finite Δ. No transfer reverses or equalizes the ordering. ■

Lemma 3 (Failure of State Sufficiency)

Let s_t be a state variable that does not encode the opportunity correspondence. Then s_t is not preference-sufficient.

Proof.

If s_t fails to encode F_t, there exist histories $h,\ h'$ such that

$$s_t(h) = s_t(h') \text{ but } x_t(h) \neq x_t(h').$$

If strict contraction holds for one of these histories, opportunity monotonicity implies a strict preference ordering between them.

Preference sufficiency would require identical continuation preferences for identical states, contradicting strict ordering.
Hence s_t cannot be preference-sufficient. ■

Corollary (Minimality)

The opportunity correspondence F_t is minimally welfare-sufficient.
Any state strictly coarser than F_t fails preference sufficiency by Lemma 3. ■

4.4 Interpretation

The failure identified in Theorem 1 is structural rather than behavioral.
Preferences remain complete and transitive.
No appeal to time inconsistency, non-expected utility, or interpersonal comparability is required. In contrast to impossibility results in social choice theory that hinge on aggregation axioms (Arrow 1951), the present boundary arises at the level of feasibility geometry itself. Two implications follow.
First, compensatory reasoning collapses because transfers cannot recreate eliminated feasible paths. Compensation presumes reversibility of opportunity sets; strict contraction violates that presumption.
Second, recursive representation collapses because state compression necessarily merges histories that differ in feasibility structure. Any representation omitting opportunity sets loses welfare-relevant information.
The boundary therefore precedes questions of curvature, discounting, or social aggregation. It is a feasibility-based constraint on representability itself.

4.5 Discussion

Theorem 1 does not assert that recursive or additive welfare frameworks are invalid in general. It identifies the precise condition under which they cannot jointly satisfy opportunity monotonicity and transfer neutrality.
The result is diagnostic: it identifies the feasibility condition under which dynamic welfare representation becomes impossible. In this sense, the theorem delineates a representational boundary analogous to domain restrictions in intertemporal aggregation (Basu and Mitra 2003; Fleurbaey and Michel 2003), but grounded in feasibility contraction rather than preference admissibility.
The next section shows that this threshold is not exceptional, since strict opportunity contraction arises generically.

5. Genericity of Strict Opportunity Contraction

The impossibility established in Theorem 1 depends on the occurrence of strict opportunity contraction. We now show that such contraction is not exceptional. Under mild feasibility conditions on dynamic feasibility, strict opportunity contraction arises generically.

The argument proceeds in three steps:

(i) defining a topology on feasible continuation correspondences,

(ii) establishing openness of strict contraction, and

(iii) establishing residuality within a natural class of dynamically productive environments.

5.1 Topological Setup

Let $\mathcal{H}_t$ denote the set of partial histories at date t.

Let $\mathcal{F}$ denote the collection of feasible continuation correspondences

$$F_t\colon\ \mathcal{H}_t \rightrightarrows \mathcal{H}.$$

We endow $\mathcal{F}$ with the topology of pointwise set inclusion stability.

Concretely, a sequence $F^n \to F$ if for every h_t and every continuation $h' \in F_t(h_t)$, membership is eventually preserved under small perturbations of the correspondence.

This topology parallels the correspondence topologies used in general equilibrium and genericity arguments (Debreu 1970; Mas-Colell 1974), where structural properties are evaluated under perturbations of feasibility mappings.

5.2 Strict Opportunity Contraction

Strict opportunity contraction at date $t$occurs if there exist histories $h,\ h'$ such that

$$F_t(h') \subsetneq F_t(h).$$

Define

$$\mathcal{C} = \{F \in \mathcal{F}\colon \exists t, \exists h, h' \text{ with } F_t(h') \subsetneq F_t(h)\}.$$

We study the topological properties of $\mathcal{C} \subset \mathcal{F}$.

5.3 Openness

Proposition 1 (Openness).

If $F \in \mathcal{C}$, then there exists an open neighborhood $\mathcal{N}(F) \subset \mathcal{F}$ such that every $\tilde{F} \in \mathcal{N}(F)$ also belongs to $\mathcal{C}$.

Proof (Sketch).

Strict inclusion implies the existence of at least one continuation h^* such that

$$h^* \in F_t(h) \text{ and } h^* \notin F_t(h').$$

Under sufficiently small perturbations that preserve membership of h^* in $F_t(h)$ and non-membership in $F_t(h')$, strict inclusion persists.

Such perturbations form an open neighborhood under the correspondence topology.

Therefore, $\mathcal{C}$ is open in $\mathcal{F}$. ■

5.4 Density in Dynamically Productive Environments

Openness does not by itself establish genericity. We therefore examine density properties.

Let $\mathcal{F}^+ \subset \mathcal{F}$ denote the subclass of dynamically productive environments, defined by the existence of actions that strictly expand reachable continuation sets on an open subset of states.

Formally, suppose the state dynamics satisfy

$$s_{t+1} = f(s_t, a_t),$$

where f is continuous in s_t and strictly increasing in a_t on an open subset $U \subset S$.

Proposition 2 (Density).

Within $\mathcal{F}^+$, the set $\mathcal{C} \cap \mathcal{F}^+$ is dense.

Proof (Sketch).

If no strict contraction exists at some $F \in \mathcal{F}^+$, then for every pair of histories the induced continuation sets are identical or weakly nested.

Because productive actions strictly expand reachable sets on U, arbitrarily small perturbations of the action set that remove such productive actions along one history generate proper inclusion.

Hence, for every neighborhood of F, there exists $\tilde{F} \in \mathcal{C} \cap \mathcal{F}^+$.

Thus strict contraction is dense in $\mathcal{F}^+$. ■

5.5 Genericity Theorem

We now combine openness and density.

Theorem 2 (Generic Strict Opportunity Contraction).

Within the class $\mathcal{F}^+$ of dynamically productive environments endowed with the correspondence topology, the set $\mathcal{C} \cap \mathcal{F}^+$ contains a nonempty open subset and is residual within that subset.

If $\mathcal{F}^+$ is a Baire space, then strict opportunity contraction holds generically in the topological sense.

5.6 Interpretation

Genericity here means that strict opportunity contraction holds on a comeager subset of dynamically productive feasibility correspondences. It is therefore structurally stable under perturbations.

The representation boundary identified in Theorem 1 is not confined to finely tuned parameterizations. It characterizes a robust region of the feasibility space.

The result aligns with the generic properties literature in economic theory (Debreu 1970; Mas-Colell 1974), but the property studied here concerns contraction of feasible continuation correspondences rather than equilibrium multiplicity or transversality.

6. Failure of Bellman Recursion

Dynamic programming rests on the premise that continuation values can be summarized by a sufficient state variable. Welfare is then represented recursively by a Bellman equation of the form

$$V(s_t) = \max_{a_t} \{u(s_t, a_t) + \beta V(s_{t+1})\},$$

where s_t compresses all payoff-relevant information contained in the history, as in the foundational recursive frameworks of Bellman (1957), Koopmans (1960), and Stokey and Lucas (1989). These frameworks rely on the compression of histories into payoff-relevant state variables satisfying a Markov sufficiency condition.

This section shows that strict opportunity contraction invalidates such recursive representation whenever the state variable fails to encode feasibility structure.

6.1 Informational Compression and Feasibility

Recursive representation presumes that histories assigned the same state yield the same continuation problem. Formally, if

$$s_t(h) = s_t(h'),$$

then the induced feasible continuation sets must coincide:

$$x_t(h) = x_t(h')$$

or equivalently $F_t(h_t) = F_t(h'_t)$.

However, under strict opportunity contraction, two histories may share identical observable states while differing in their feasible continuation sets, so that

$$s_t(h) = s_t(h') \text{ but } x_t(h) \neq x_t(h').$$

If the state variable omits feasibility structure, it compresses distinct opportunity correspondences into a single informational state. Under opportunity monotonicity, such compression conflicts with welfare ordering.

6.2 Bellman Inconsistency

We formalize this breakdown.

Proposition 3 (Failure of Bellman Sufficiency)

Suppose strict opportunity contraction occurs at date t. If a state variable s_t is not rich enough to distinguish histories with different opportunity correspondences $F_t(h_t)$, then no Bellman equation defined on s_t can represent preferences satisfying A1.

This result identifies a structural violation of the recursive representation conditions characterized in Koopmans (1960). Whereas Koopmans derives existence of recursive representation from stationarity and separability, the present argument shows that feasibility contraction destroys the informational compression required for recursion.

Proof (Sketch).

If $s_t(h) = s_t(h')$ but $x_t(h) \neq x_t(h')$, equivalently $F_t(h_t) \neq F_t(h'_t)$, the Bellman representation assigns identical continuation values:

$$V(s_t(h)) = V(s_t(h')).$$

Under strict contraction, opportunity monotonicity implies a strict preference ordering between the two histories.

The recursive representation therefore contradicts the preference ordering. ■

The inconsistency arises not because stationarity or separability is violated, but because the state variable is informationally insufficient. The Bellman equation presumes that continuation problems are identical whenever states coincide. Strict contraction violates that presumption.

6.3 Dynamic Programming as Implicit Domain Restriction

The preceding result clarifies the implicit domain restriction underlying recursive methods.

Dynamic programming is valid when feasible continuation sets are fully determined by the state variable. In recursive macroeconomic and growth models, this condition is typically satisfied because technological and resource constraints evolve according to a law of motion fully captured by the state (Stokey and Lucas 1989). In such domains, the opportunity correspondence $F_t(h_t)$, or equivalently the induced set $x_t(h)$, is a function of $s_t(h)$, so informational compression preserves welfare-relevant structure. The present analysis highlights that when feasibility depends on irreversible opportunity loss not encoded in s_t, the Markov structure collapses. Strict opportunity contraction lies outside that domain.

The result does not reject recursive methodology; it delineates its domain of structural applicability. It identifies the feasibility condition under which its foundational assumption fails.

6.4 Illustration: Feasibility versus Wealth

To illustrate the distinction, consider a setting in which two histories yield identical wealth levels at date t, but one history involves irreversible loss of training or quality, thereby eliminating some future feasible actions. Figure 1 illustrates this distinction.

A wealth-based state variable would treat the histories as identical. Yet the feasible continuation sets differ.

Under opportunity monotonicity, the history preserving feasible actions is strictly preferred, even if wealth is equalized through transfers.

The Bellman equation defined solely on wealth cannot represent this ordering.

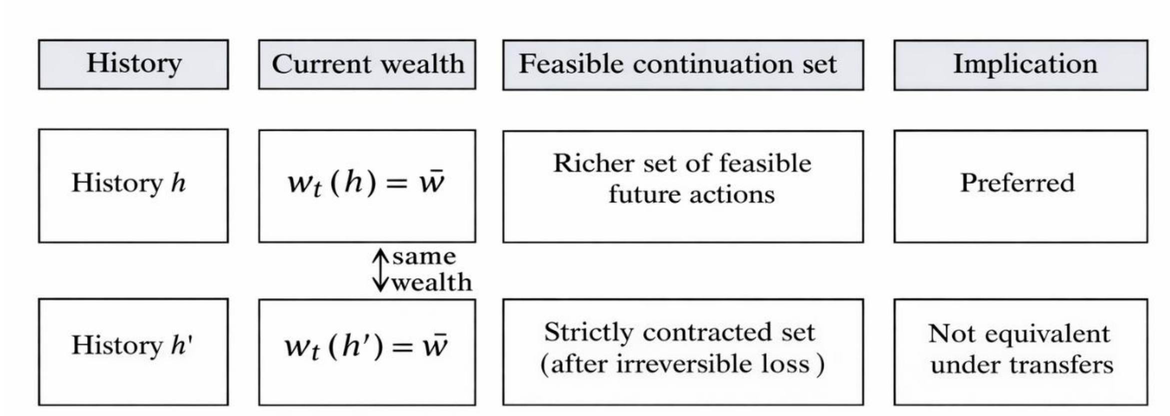

Figure 1. Feasibility versus Wealth

6.5 Structural Interpretation and Relation to the Representation Boundary

The failure of Bellman recursion reinforces the structural interpretation of Theorem 1.

The result does not contradict recursive dynamic theory; it delineates the domain within which recursive compression remains valid. Outside that domain—when strict opportunity contraction is admitted—no state compression consistent with opportunity monotonicity is possible.

Recursive representation relies on the possibility of compressing dynamic feasibility into a lower-dimensional state space. Strict opportunity contraction introduces irreducible heterogeneity in feasible continuation sets.

Any state variable omitting that structure collapses welfare-relevant distinctions.

Thus, representation failure in this context is not due to preference anomalies or behavioral departures from rationality. It arises from the geometry of feasible sets under irreversibility.

Theorem 1 establishes the minimality of the opportunity correspondence as the welfare-sufficient state. Proposition 3 shows that Bellman recursion fails whenever the chosen state variable is strictly coarser than that minimal object.

The breakdown of recursive representation is therefore a corollary of the structural boundary identified earlier.

Accordingly, Theorem 1 can be interpreted as a feasibility-based counterpart to recursive representation theorems: rather than deriving recursive structure from axioms, it derives the impossibility of recursion from structural contraction.

The next section extends the boundary from individual evaluation to social aggregation.

7. Social Extension

The structural boundary identified at the individual level extends naturally to social evaluation. When feasible continuation sets contract for at least one individual without offsetting expansion for others, compensability and state-sufficient representation fail at the social level as well.

The social extension reveals that feasibility contraction operates as a pre-aggregation constraint: impossibility arises before distributive or fairness considerations enter.

7.1 Individual Opportunity Correspondences

Let I be a finite set of individuals.

Each individual $i \in I$ is associated with an opportunity correspondence

$$F_t^i(h_t),$$

and we write

$$x_t^i(h) := F_t^i(h_t)$$

for notational consistency.

Define the product opportunity correspondence

$$F_t(h_t) = \left(F_t^1(h_t), \dots, F_t^{|I|}(h_t)\right).$$

This object records the feasible continuation sets for all individuals simultaneously.

7.2 Social Preferences

Let social preferences over histories be complete and transitive.

We impose a natural extension of opportunity monotonicity.

A1-S (Pareto Opportunity Monotonicity)

If, for all individuals i,

$$x_t^i(h) \supseteq x_t^i(h'),$$

and for at least one individual the inclusion is strict, then

$$h \succ h'.$$

Strict expansion of feasible continuation sets for at least one individual, without contraction for others, strictly increases social welfare.

We also maintain transfer neutrality at the social level: finite transfers do not expand feasible continuation sets.

7.3 Social Representation Boundary

We now state the social analogue of Theorem 1.

Theorem 3 (Social Representation Impossibility)

Suppose social preferences satisfy Pareto Opportunity Monotonicity and Transfer Neutrality.

If strict opportunity contraction occurs for at least one individual at date t, and no individual experiences strict expansion, then:

(i) No finite transfer restores social welfare equivalence; and

(ii) No state variable that fails to encode the product opportunity correspondence F_t is preference-sufficient for social evaluation.

Consequently, the product opportunity correspondence constitutes the minimal welfare-sufficient social state.

7.4 Proof Sketch

If for some individual i,

$$x_t^i(h') \subset x_t^i(h),$$

and for all others inclusion is weak, Pareto opportunity monotonicity implies

$$h \succ h'.$$

Transfer neutrality preserves the contraction under finite transfers. Hence no transfer equalizes the social ranking.

If a state variable fails to encode F_t, there exist histories with identical states but distinct product opportunity correspondences. Pareto opportunity monotonicity then implies distinct welfare rankings, contradicting preference sufficiency. ■

7.5 Structural Interpretation

The social extension highlights two features of the representation boundary.

First, compensatory reasoning at the social level inherits the same structural limitation as at the individual level. Transfers redistribute resources but cannot recreate eliminated feasible continuation sets.

Second, informational compression becomes even more demanding under aggregation. A social state variable must encode the joint feasibility structure across individuals. Any coarser representation collapses socially relevant distinctions.

The boundary therefore persists under aggregation. It does not depend on interpersonal comparability, cardinal utilities, or distributive weights. It arises from structural properties of feasible continuation correspondences.

7.6 Relation to Domain-Based Results

The result differs fundamentally from aggregation-based impossibility theorems in social choice theory. Rather than arising from conflicts among axioms governing interpersonal aggregation—as in the social-choice frameworks of Arrow (1951), Sen (1970), and Hammond (1976)—the boundary identified here arises prior to aggregation and is driven by feasibility contraction itself. Social preferences may be complete, transitive, and satisfy Pareto opportunity monotonicity; the failure does not rely on interpersonal comparability, non-dictatorship, or independence conditions.

It arises from the structural geometry of feasible continuation sets. Once strict opportunity contraction occurs for at least one individual without offsetting expansion for others, compensatory reasoning and state-sufficient representation collapse at the social level, regardless of the aggregation rule employed. It isolates a feasibility-based constraint on representability that operates independently of classical aggregation paradoxes.

8. Market Implementation

The structural boundary identified above is not confined to abstract feasibility structures. It can arise in equilibrium in standard market environments.

This section illustrates how strict opportunity contraction may be implemented through quality adjustments under binding regulation.

8.1 Environment

Consider a regulated market with a binding price ceiling $\bar{p}$.

Firms choose product quality $q \in Q \subset \mathbb{R}_+$.

Demand depends on quality, $D(q)$, with $D'(q) \geq 0$.

Production cost is $c(q)$, with $c'(q) > 0$.

Consumers' feasible continuation sets depend on quality. Let

$$F_t(q)$$

denote the opportunity correspondence induced by quality q.

Assume:

$$q' > q \Rightarrow F_t(q') \supseteq F_t(q),$$

with strict inclusion on a nonempty open subset of Q.

Higher quality preserves or expands feasible continuation opportunities.

8.2 Equilibrium Quality under Regulation

Under the binding price ceiling, firms solve

$$\max_{q \in Q} \ \pi(q; \bar{p}) = \bar{p} \cdot D(q) - c(q).$$

Let q^* denote the equilibrium quality satisfying the first-order condition under the binding constraint.

Let q^{FB} denote the first-best quality that would maximize welfare absent the price ceiling.

Under standard regularity conditions, a binding price ceiling implies

$$q^* < q^{FB}.$$

Regulation thus induces quality distortion. Quality distortions under binding price ceilings are well documented in regulatory theory (Spence 1975; Laffont and Tirole 1993). In regulated environments where prices cannot adjust upward, firms optimally substitute away from price adjustment toward non-price margins such as quality, maintenance, or service intensity. The distortion is not accidental but arises as an equilibrium response to binding price constraints.

When product quality affects consumers' feasible continuation sets—such as through time burdens, reliability, or access to complementary actions—a reduction in quality contracts the opportunity correspondence. In this way, regulatory quality distortion provides a concrete market mechanism through which strict opportunity contraction may arise in equilibrium.

8.3 Opportunity Contraction in Equilibrium

If strict inclusion holds on an open region of Q, then

$$F_t(q^*) \subset F_t(q^{FB})$$

for some date t.

Equilibrium quality therefore induces strict opportunity contraction relative to the social optimum.

By Theorem 1, this contraction implies:

1. No finite transfer restores welfare equivalence between equilibrium and social optimum.
2. No state variable omitting the opportunity correspondence suffices for representation.

8.4 Structural Interpretation and Generalization

The mechanism is structural.

The price ceiling prevents firms from adjusting prices. Quality becomes the margin of adjustment.

Reduced quality shifts time burdens or eliminates feasible future actions for consumers—such as access to timely services, productive investments, or skill formation. Transfers may equalize wealth ex post but cannot recreate eliminated feasible actions. The contraction lies in feasibility, not income.

Representation failure thus arises within standard competitive or regulated environments. It is not an artifact of exotic preferences or nonstandard markets.

The quality-distortion example is illustrative rather than exhaustive. Strict opportunity contraction may arise whenever feasibility depends on prior investment or quality, time-sensitive actions generate future opportunities, and regulatory or technological constraints eliminate feasible paths. The essential feature is the dependence of feasible continuation sets on irreversible decisions. Market environments provide a concrete realization of this structure.

The market implementation also clarifies that the representation boundary is not purely conceptual. It can emerge endogenously in equilibrium when regulatory constraints alter feasibility structures. The impossibility result therefore has practical relevance: even in standard policy environments, compensatory transfers and recursive representation may fail to capture welfare-relevant losses.

9. Conclusion

This paper identifies a structural boundary of dynamic welfare representation. Under opportunity monotonicity and transfer neutrality, strict contraction of feasible continuation sets implies a joint failure of finite compensability and state-sufficient recursive representation. The core point is nonparametric. The impossibility does not depend on discounting, additivity, curvature, or interpersonal aggregation. It arises because dynamic welfare comparisons depend not only on realized payoffs, but also on which future actions remain feasible.

The underlying logic is straightforward. When irreversible time loss removes feasible continuations, no finite transfer can reconstruct the lost opportunity set. Compensation therefore fails, not because transfers are insufficiently large in a quantitative sense, but because feasibility itself has changed. Once this occurs, any state description that omits the opportunity correspondence collapses welfare-relevant distinctions. Recursive sufficiency fails for structural rather than merely behavioral reasons. In this sense, the paper reverses the usual direction of representation analysis: instead of deriving recursive evaluation from axiomatic regularity, it isolates a feasibility configuration under which recursive representation must break down.

The paper also shows that this boundary is not an exceptional curiosity. Strict opportunity contraction arises generically in dynamically productive environments, and the same mechanism extends beyond the baseline theorem. Bellman recursion inherits the same fragility when feasibility is omitted from the state. Social evaluation does not escape the problem, since aggregation cannot restore opportunities that irreversibility has removed. Regulated market environments with endogenous quality distortion provide an implementation channel through which the same structural logic appears in equilibrium.

Taken together, these results locate irreversibility as a representational boundary that is logically prior to discounting, curvature, or aggregation. The contribution of this discussion paper version is to make that logical architecture explicit across recursive, social, and market domains.

Appendix

Appendix A. Full Proof of Theorem 1

This appendix provides a complete proof of Theorem 1.

A.1 Persistence of Strict Contraction

Let strict opportunity contraction hold at date t. Then there exist histories $h,\ h'$ such that

$$F_t(h'_t) \subset F_t(h_t).$$

By Transfer Neutrality (A2), for any finite transfer Δ,

$$F_t((h' + \Delta)_t) = F_t(h'_t).$$

Hence,

$$F_t((h' + \Delta)_t) \subset F_t(h_t)$$

for all finite Δ. ■

A.2 Failure of Finite Compensability

By Opportunity Monotonicity (A1), strict inclusion implies

$$h \succ h'.$$

By A.1, strict inclusion persists under finite transfers. Therefore,

$$h \succ h' + \Delta \text{for all finite } \Delta.$$

Hence no finite transfer restores welfare equivalence. ■

A.3 Failure of Preference Sufficiency

Let s_t be a state variable that does not encode the opportunity correspondence.

Then there exist histories $h,\ h'$ such that

$$s_t(h) = s_t(h') \text{ but } F_t(h_t) \neq F_t(h'_t).$$

If strict contraction holds between the two histories, A1 implies strict preference ordering.

Preference sufficiency would require identical continuation evaluation for identical states, contradicting strict ordering.

Hence s_t cannot be preference-sufficient.

A.4 Minimality

Suppose a strictly coarser state variable $\tilde{s}_t$ exists.

Then there exist histories with distinct opportunity correspondences but identical $\tilde{s}_t$. By A.3, $\tilde{s}_t$ fails preference sufficiency.

Therefore the opportunity correspondence is minimally welfare-sufficient. ■

Appendix B. Genericity of Strict Opportunity Contraction

This appendix formalizes the genericity argument.

B.1 Topological Structure

Let $\mathcal{X}$ denote the space of feasible continuation correspondences.

Endow $\mathcal{X}$ with the topology induced by pointwise inclusion under finite perturbations of feasible sets.

A sequence $F_t^n \to F_t$ if, for each partial history, inclusion relations stabilize under perturbations.

B.2 Openness

Suppose strict contraction holds:

$$F_t(h_t') \subset F_t(h_t).$$

Then there exists at least one continuation $\bar{h} \in F_t(h_t) \setminus F_t(h_t')$.

Under sufficiently small perturbations that preserve membership of $\bar{h}$ in the first set and exclusion from the second, strict inclusion persists.

Thus the set of correspondences exhibiting strict contraction contains a nonempty open subset.

B.3 Density

Consider any dynamically productive environment in which actions expand feasible continuation sets.

Small perturbations that eliminate productive action at a date create strict contraction relative to the unperturbed environment.

Hence for any neighborhood in $\mathcal{X}$, one can construct a correspondence exhibiting strict contraction.

The set of such correspondences is dense.

B.4 Residuality

Because strict contraction holds on a nonempty open set and dense subsets can be constructed in dynamically productive classes, the set of correspondences exhibiting strict contraction is residual within that open subset.

Therefore strict opportunity contraction is generic in the Baire category sense. ■

Appendix C. Bellman Inconsistency

Let a Bellman equation be defined on state space $\mathcal{S}$:

$$V(s_t) = \max_{a_t} \{u(s_t, a_t) + \beta V(s_{t+1})\}.$$

Suppose strict opportunity contraction holds and s_t fails to encode F_t.

Then there exist h, h' such that

$$s_t(h) = s_t(h') \text{ but } F_t(h_t) \neq F_t(h_t').$$

The Bellman equation assigns identical continuation values, contradicting opportunity monotonicity.

Hence recursive representation fails whenever feasibility is not encoded in the state. ■

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