Physics 364: Problem Set 3

Sean Carroll, Winter 2005

Due Wednesday 2 February, 12:00 noon.

1. (20 points) Given a vector n^μ normalized to $g_{\mu\nu}n^\mu n^\nu = \sigma$ (where $\sigma = \pm 1$ depending on whether n^μ is spacelike or timelike), we can construct the tensor

$$P^\mu{}_\nu = \delta^\mu{}_\nu - \sigma n^\mu n_\nu \ .$$

This is known as the "first fundamental form on the submanifold orthogonal to n."

   (a) Show that $P^\mu{}_\nu$ projects a vector A^μ into one orthogonal to n^μ; that is, show that $P^\mu{}_\nu A^\nu$ is both orthogonal to n^μ and also unaffected by P:

$$P^\mu{}_\nu P^\nu{}_\rho A^\rho = P^\mu{}_\sigma A^\sigma \ .$$

   (b) Show that $P_{\mu\nu}$ acts like the metric on vectors orthogonal to n; that is, for A^μ and B^ν orthogonal to n,

$$P_{\mu\nu}A^\mu B^\nu = g_{\mu\nu}A^\mu B^\nu \ .$$

   (c) In a 3-dimensional Minkowski space with coordinates $\{t, x, y\}$, consider the timelike hyperboloid described by

$$x^2 + y^2 = t^2 + 1 \ .$$

   Find the components $\{n^t, n^x, n^y\}$ of the vector field orthogonal to the hyperboloid. (Possibly, although not necessarily, useful hint: the hyperboloid is left invariant under Lorentz transformations.) You can express the components as functions of angular coordinates $\{r, \theta\}$ in space if you like (although you want $\{n^t, n^x, n^y\}$, not $\{n^t, n^r, n^\theta\}$).

   (d) Calculate $P_{\mu\nu}$ for the hyperboloid.

2. (10 points) Section 2.6 of the book explores the metric of an expanding universe with flat spatial slices,

$$ds^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}x^2 + \mathrm{d}x^2 + \mathrm{d}x^2] \ ,$$

and shows that for $a(t) \propto t^q$, with $0 < q < 1$, there can be "horizons" – we can find two events with spacelike separation such that their past light cones do not overlap. Find an example of some other form of $a(t)$ such that there are horizons to both the past and the future – we can find two events with spacelike separation such that neither the past nor the future light cones overlap.

3. (10 points) Consider R^3 as a manifold with the flat Euclidean metric, and coordinates $\{x, y, z\}$. Introduce spherical polar coordinates $\{r, \theta, \phi\}$ related to $\{x, y, z\}$ by

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \ , \end{aligned} \tag{0.1}$$

so that the metric becomes

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \ .$$

(a) A particle moves along a parameterized curve given by

$$x(\lambda) = \cos\lambda \ , \quad y(\lambda) = \sin\lambda \ , \quad z(\lambda) = \lambda \ .$$

Express the path of the curve in the $\{r, \theta, \phi\}$ system.

(b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

4. (10 points) Consider Maxwell's equations, $\mathrm{d}F = 0$ and $\mathrm{d}{*F} = {*J}$, in a two-dimensional spacetime.

(a) Explain why one of the two equations can be discarded.

(b) Show that the electromagnetic field can be expressed in terms of a scalar field, and write out the field equations for this scalar in component form.