Physics 364: Problem Set 6

Sean Carroll, Winter 2005

Due Wednesday 23 February, 12:00 noon.

1. (15 pts) Consider a theory that looks something like electromagnetism, except that the potential is a two-form $B_{\mu\nu}$ instead of a one-form A_μ. Let the field strength be a three-form $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$. The action for the theory is given by

$$S = \int d^4x \sqrt{-g} \left(-H^{\mu\nu\rho} H_{\mu\nu\rho} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} J_{\rho\sigma} \right) ,$$

where $J_{\mu\nu} = -J_{\mu\nu}$ is an antisymmetric "current" two-form that is independent of the metric.

   (a) Derive the equations of motion for the field.

   (b) Derive the energy-momentum tensor.

   (c) What is the condition on $J_{\mu\nu}$ needed to ensure that the action is invariant under "gauge transformations" of the form $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu} \lambda_{\nu]}$ for an arbitrary one-form λ_ν? Are the equations of motion and energy-momentum tensor gauge-invariant?

2. (20 pts) In class we derived Einstein's equation by varying the Hilbert action with respect to the metric. It is interesting to observe that they can also be derived by treating the metric and connection as independent degrees of freedom and varying separately with respect to them; this is known as the **Palatini formalism**. That is, we consider the action

$$S = \int d^4x \sqrt{-g}\; g^{\mu\nu} R_{\mu\nu}(\Gamma) ,$$

where the Ricci scalar is thought of as constructed purely from the connection, not using the metric. From what was done in class, it is immediate that variation with respect to the metric gives the usual Einstein's equations, but for a Ricci tensor constructed from a connection which has no *a priori* relationship to the metric. Imagining from the start that the connection is symmetric (torsion free), show that variation of this action with respect to the connection coefficients leads to the requirement that the connection be metric compatible (*i.e.*, the Christoffel connection). Remember that Stokes's theorem, relating the integral of the covariant divergence of a vector to an integral of the vector over the boundary, does not work for a general covariant derivative. The best strategy is to write the connection coefficients as a sum of the Christoffel symbols $\widetilde{\Gamma}^{\lambda}_{\mu\nu}$ and a tensor ${C^{\lambda}}_{\mu\nu}$,

$$\Gamma^{\lambda}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}_{\mu\nu} + {C^{\lambda}}_{\mu\nu} \ ,$$

and then show that ${C^{\lambda}}_{\mu\nu}$ must vanish.

3. (15 pts) Consider an $(N+n+1)$-dimensional spacetime with coordinates $\{t, x^I, y^i\}$, where I goes from 1 to N and i goes from 1 to n. Let the metric be

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\delta_{IJ}\mathrm{d}x^I\mathrm{d}x^J + b^2(t)\gamma_{ij}(y)\mathrm{d}y^i\mathrm{d}y^j \ , \tag{0.1}$$

where δ_{IJ} is the usual Kronecker delta and $\gamma_{ij}(y)$ is the metric on an n-dimensional maximally symmetric spatial manifold. Imagine that we normalize the metric γ such that the curvature parameter

$$k = \frac{R(\gamma)}{n(n-1)} \tag{0.2}$$

is either $+1$, 0, or -1, where $R(\gamma)$ is the Ricci scalar corresponding to γ_{ij}.

(a) Calculate the Ricci tensor for this metric.

(b) Define an energy-momentum tensor in terms of an energy density ρ and pressure in the x^I and y^i directions, $p^{(N)}$ and $p^{(n)}$:

$$T_{00} = \rho \tag{0.3}$$
$$T_{IJ} = a^2 p^{(N)}\delta_{IJ} \tag{0.4}$$
$$T_{ij} = b^2 p^{(n)}\gamma_{ij} \ . \tag{0.5}$$

Plug the metric and $T_{\mu\nu}$ into Einstein's equations to get three independent equations for a and b.

(c) Derive equations for the energy density and the two pressures at a static solution (where $\dot{a} = \dot{b} = \ddot{a} = \ddot{b} = 0$), in terms of k, n, and N. Use these to derive expressions for the equation-of-state parameters $w^{(N)} = p^{(N)}/\rho$ and $w^{(n)} = p^{(n)}/\rho$, valid at the static solution.