# Graph algorithms, computational motifs, and GraphBLAS

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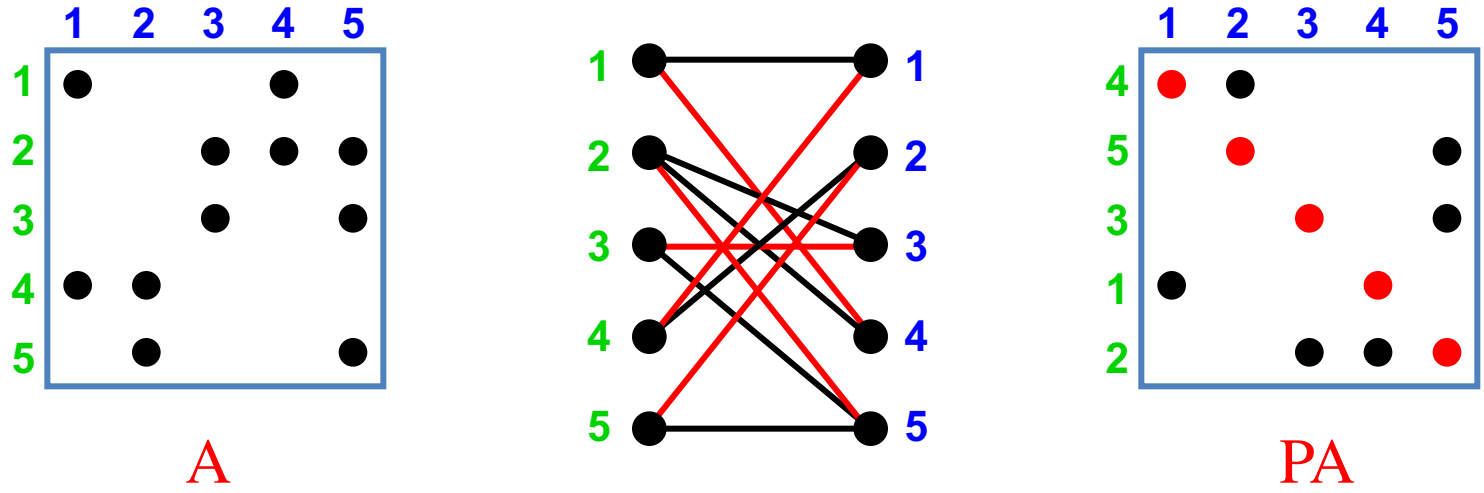
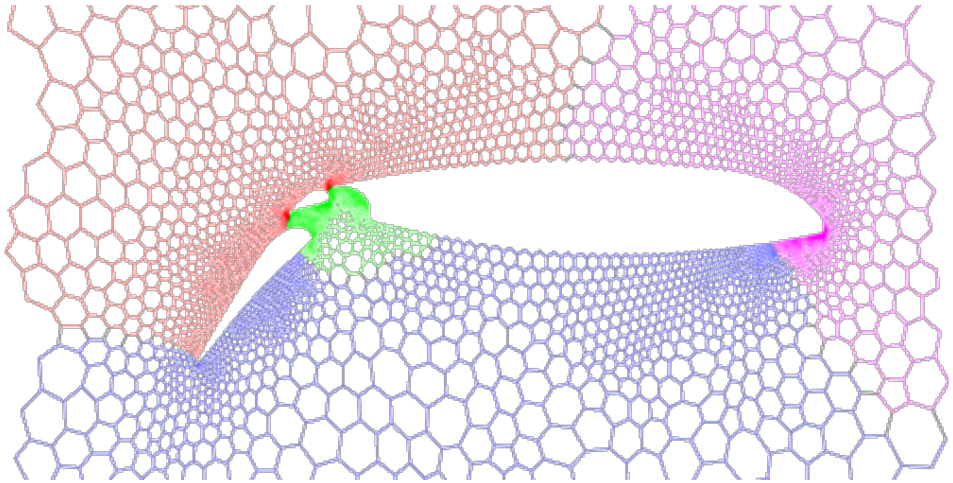
# Outline

- Motivation
- Case studies:
  A. **Graph traversals:** Breadth-first search
     - Motif: Sparse matrix times sparse vector (SpMSpV)
  B. **Maximal Independent Sets:** Luby's algorithm
     - Motif: SpMSpV
  C. **Triangle Counting**
     - Motif: SpGEMM
  D. **Betweenness Centrality (optional)**
     - Motif: SpMSpV or sparse matrix-matrix multiply (SpGEMM)

# Large Graphs in Scientific Computing




Matching in bipartite graphs: Permuting to heavy diagonal or block triangular form



**Graph partitioning:** *Dynamic load balancing* in parallel simulations

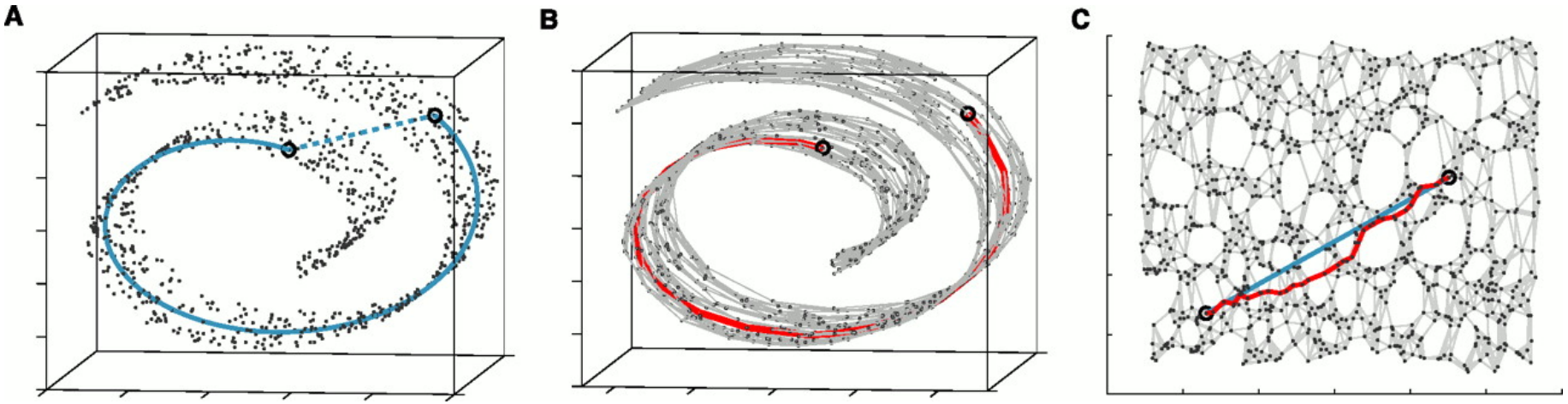
Picture (left) credit: Sanders and Schulz

**Problem size:** as big as the sparse linear system to be solved or the simulation to be performed

# Manifold Learning

**Isomap (Nonlinear dimensionality reduction):** Preserves the intrinsic geometry of the data by using the geodesic distances on manifold between all pairs of points

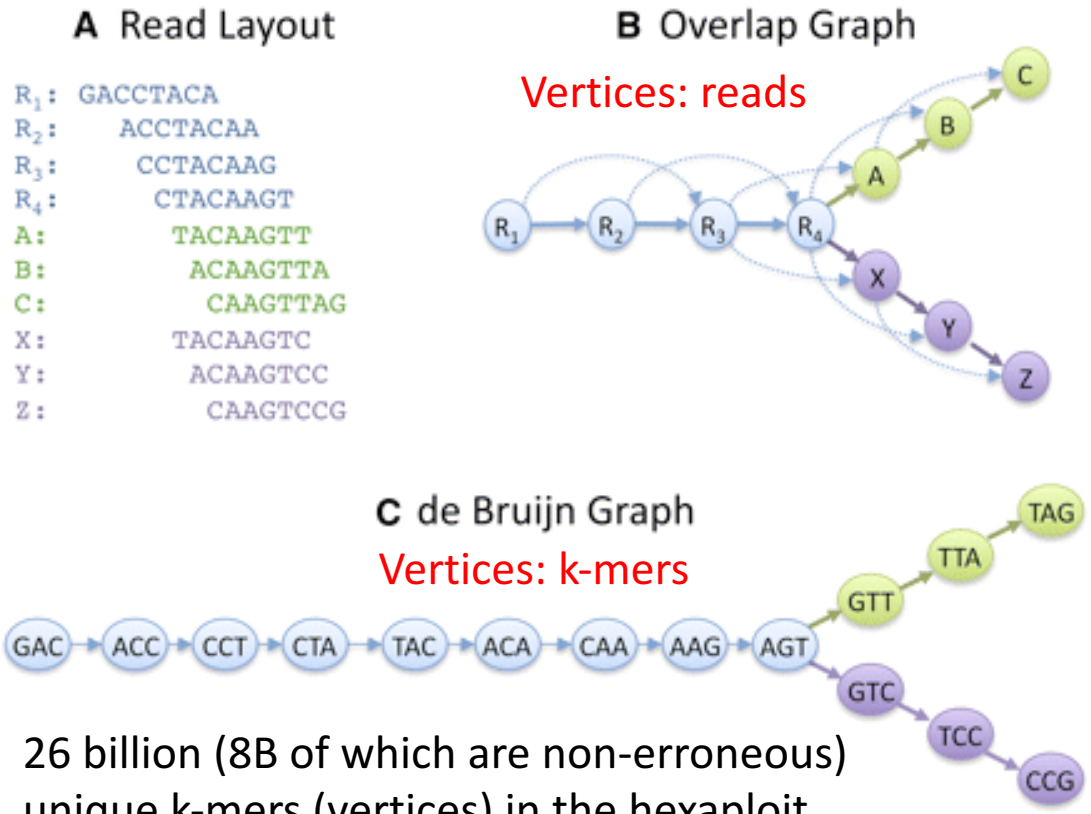
**Tools used or desired:**
- K-nearest neighbors
- ***All pairs shortest paths (APSP)***
- Top-k eigenvalues




Tenenbaum, Joshua B., Vin De Silva, and John C. Langford. "A global geometric framework for nonlinear dimensionality reduction." Science 290.5500 (2000): 2319-2323.

# Large Graphs in Biology

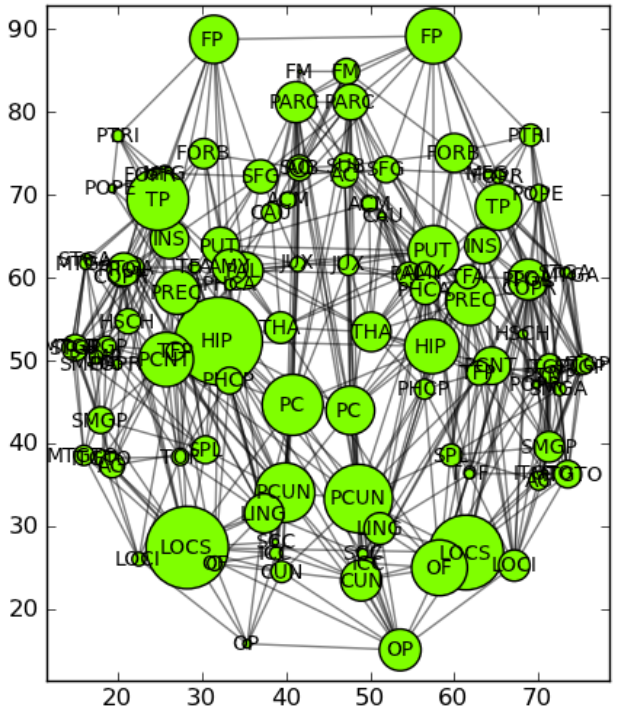
## Whole genome assembly




26 billion (8B of which are non-erroneous) unique k-mers (vertices) in the hexaploit wheat genome W7984 for k=51

Schatz et al. (2010) Perspective: Assembly of Large Genomes w/2nd-Gen Seq. Genome Res. (figure reference)

## Graph Theoretical analysis of Brain Connectivity




Potentially millions of neurons and billions of edges with developing technologies

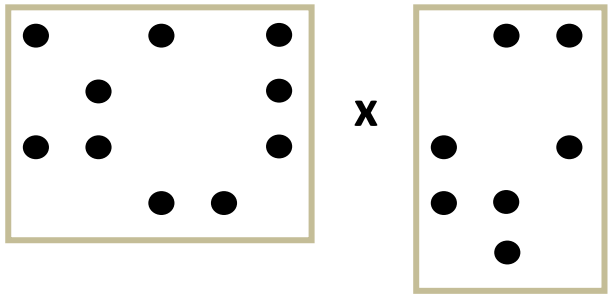
# The case for sparse matrices

Many irregular applications contain coarse-grained parallelism that can be exploited by abstractions at the proper level.

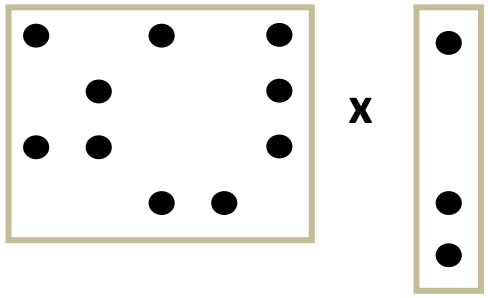
| Traditional graph computations | Graphs in the language of linear algebra |
|---|---|
| Data driven, unpredictable communication. | Fixed communication patterns |
| Irregular and unstructured, poor locality of reference | Operations on matrix blocks exploit memory hierarchy |
| Fine grained data accesses, dominated by latency | Coarse grained parallelism, bandwidth limited |

# Linear-algebraic primitives for graphs

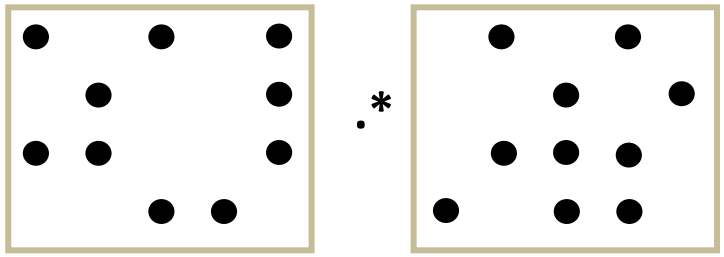
Sparse matrix X sparse matrix



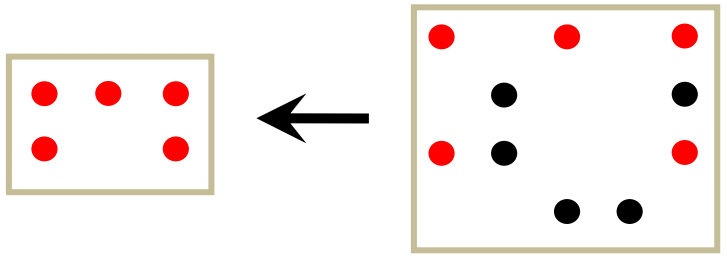
Sparse matrix X sparse vector



Element-wise operations



Sparse matrix indexing



Is **think-like-a-vertex** really more productive?

"Our mission is to build up a linear algebra sense to the extent that vector-level thinking becomes as natural as scalar-level thinking."

**- Charles Van Loan**

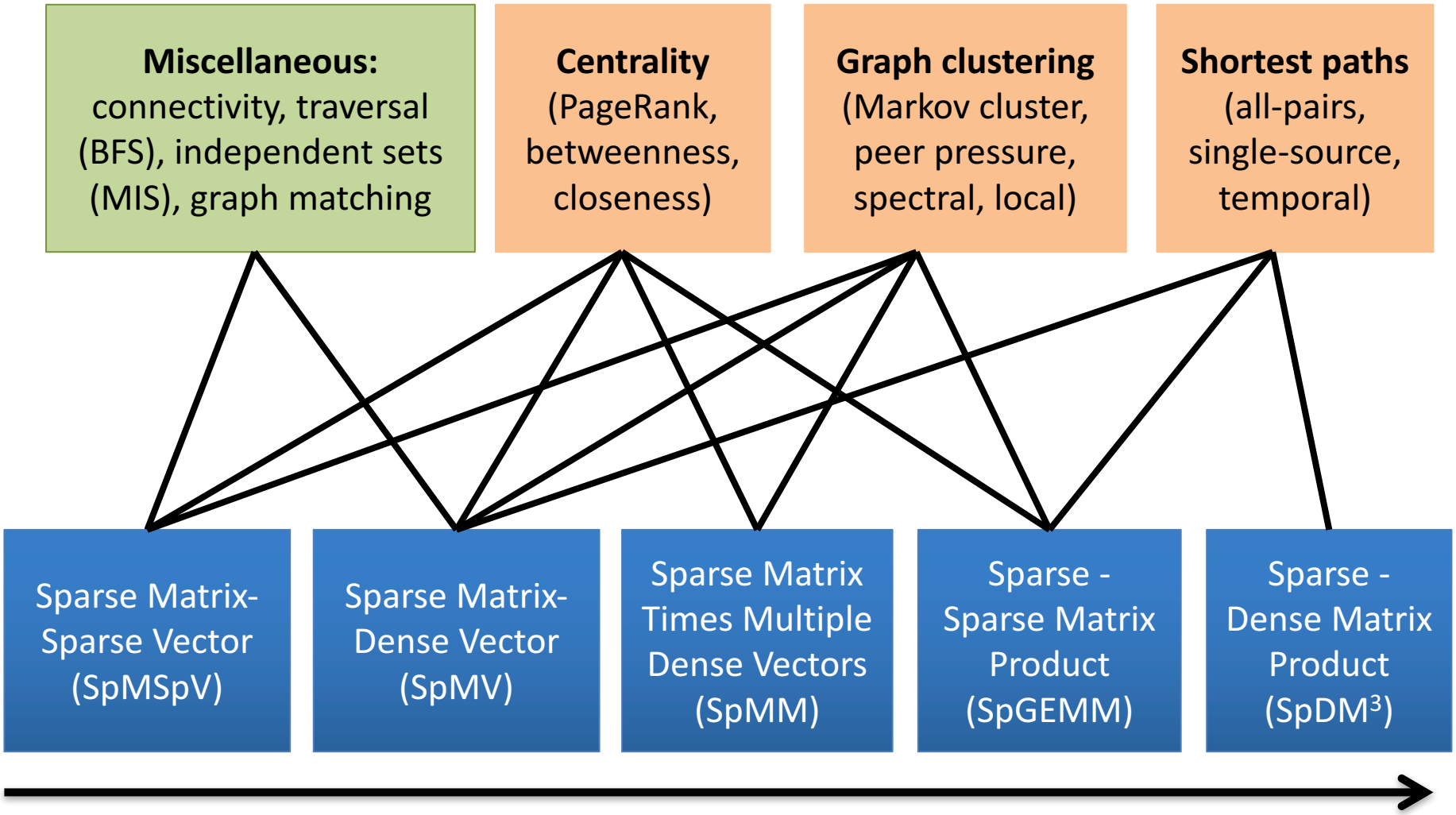
# Examples of semirings in graph algorithms

| | |
|---|---|
| Real field: **(R, +, x)** | Classical numerical linear algebra |
| Boolean algebra: **({0 1}, \|, &)** | Graph connectivity |
| Tropical semiring: **(R U {∞}, min, +)** | Shortest paths |
| **(S, select, select)** | Select subgraph, or contract nodes to form quotient graph |
| (edge/vertex attributes, vertex data aggregation, edge data processing) | Schema for user-specified computation at vertices and edges |
| **(R, max, +)** | Graph matching &network alignment |
| **(R, min, times)** | Maximal independent set |

- **Shortened semiring notation: (Set, Add, Multiply)**. Both identities omitted.
- **Add:** Traverses edges, **Multiply:** Combines edges/paths at a vertex
- Neither add nor multiply needs to have an inverse.
- Both **add** and **multiply** are **associative**, **multiply distributes** over **add**

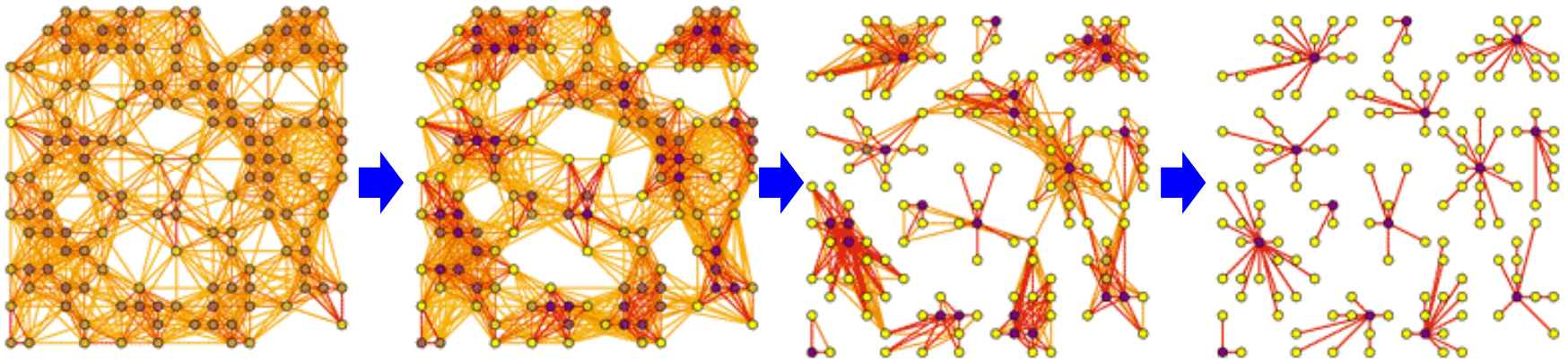
# Graph Algorithms on GraphBLAS

http://graphblas.org



# Markov Cluster Algorithm (MCL)

Widely popular and successful algorithm for discovering clusters in protein interaction and protein similarity networks




**At each iteration:**

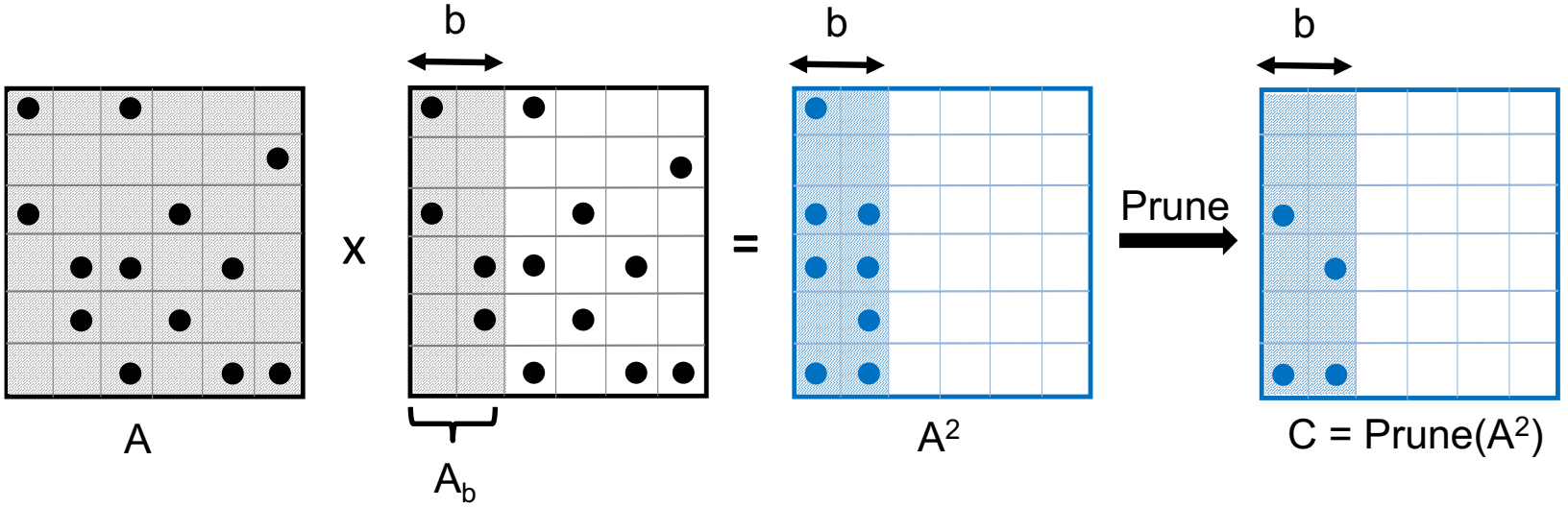
**Step 1** (Expansion): Squaring the matrix while pruning (a) small entries, (b) denser columns

**Naïve implementation:** sparse matrix-matrix product (SpGEMM), followed by column-wise top-K selection and column-wise pruning

**Step 2** (Inflation) : taking powers entry-wise

# HipMCL: High-performance MCL (ExaBiome)

MCL process is both **computationally expensive** and **memory hungry**, limiting the sizes of networks that can be clustered

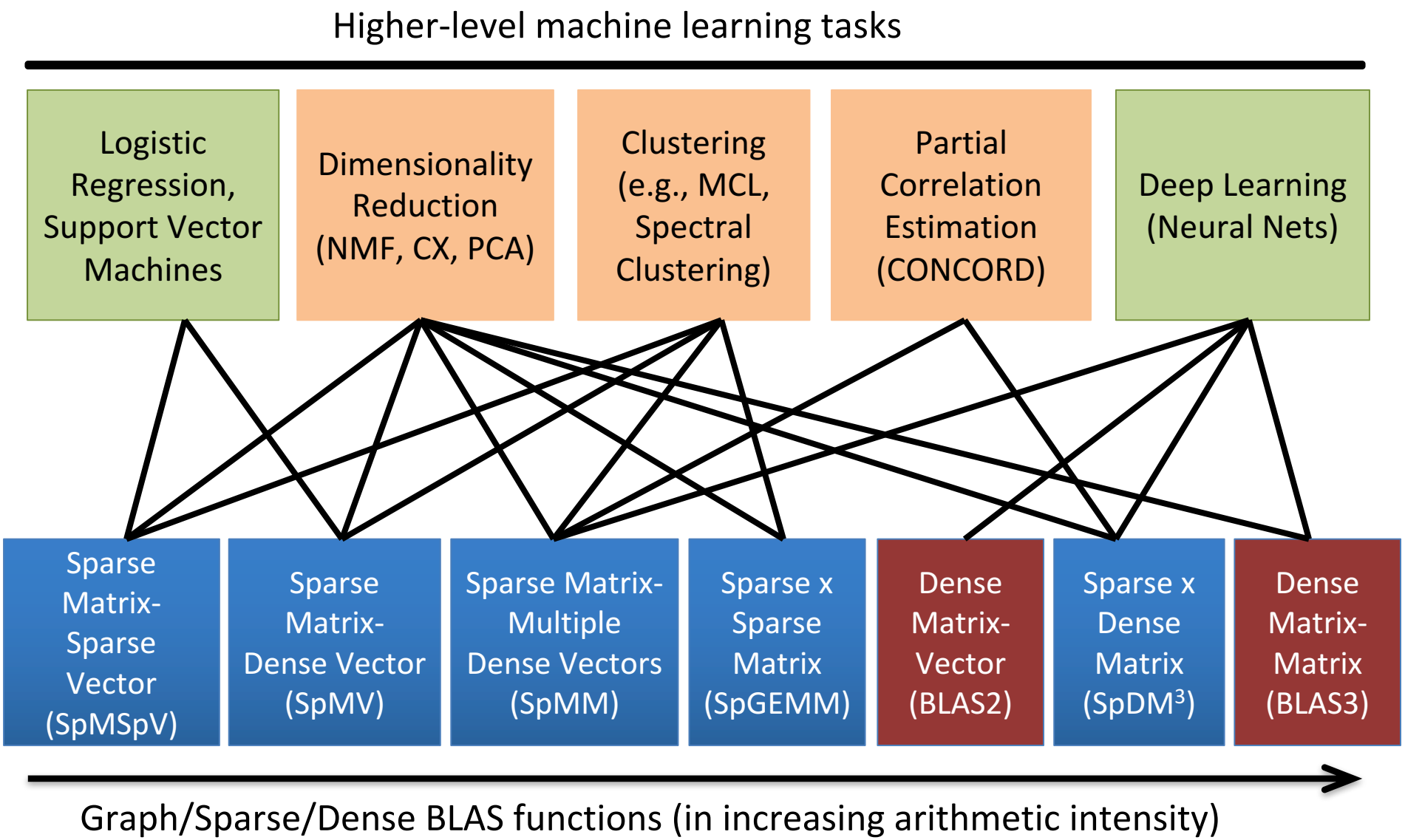



- HipMCL overcomes such limitation via **sparse parallel algorithms**.
- **Up to 1000X times faster** than original MCL with same accuracy.
- Easily clusters a network of ~75M nodes with ~68B edges in ~2.4 hours using ~2000 nodes of Cori/NERSC.

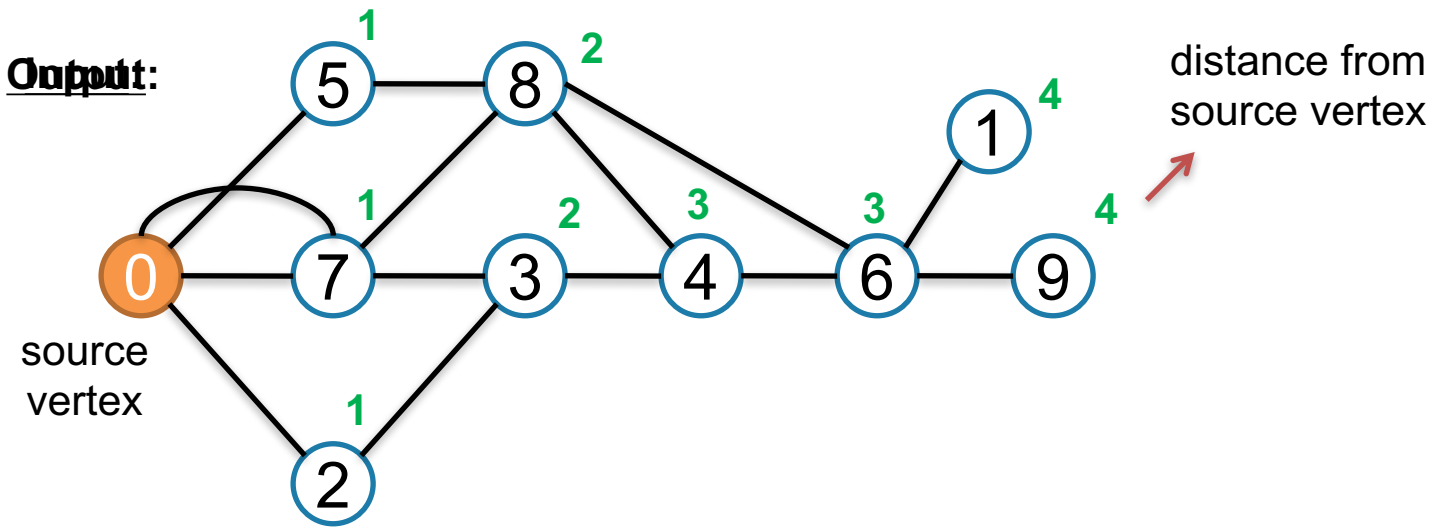
A. Azad, G. Pavlopoulos, C. Ouzounis, N. Kyrpides, A. Buluç; HipMCL: a high-performance parallel implementation of the Markov clustering algorithm for large-scale networks, *Nucleic Acids Research, 2018*

# Machine Learning on [Graph]BLAS

http://graphblas.org



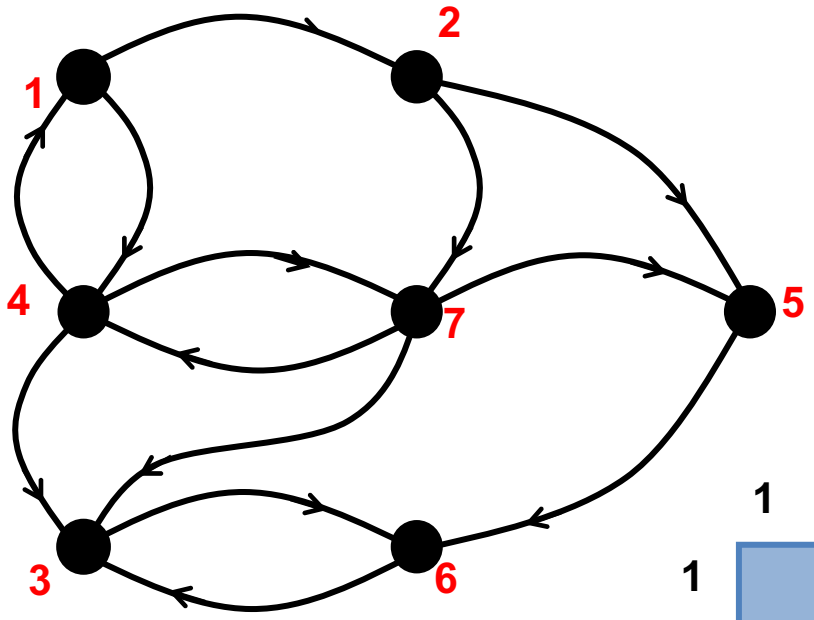
# Graph traversal : Breadth-first search (BFS)



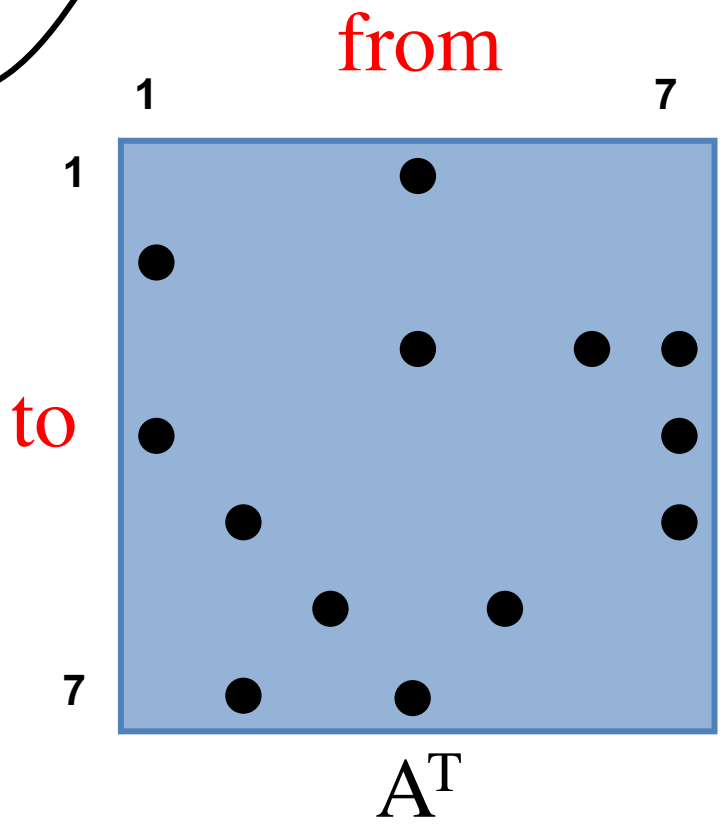

**Memory requirements (# of machine words):**

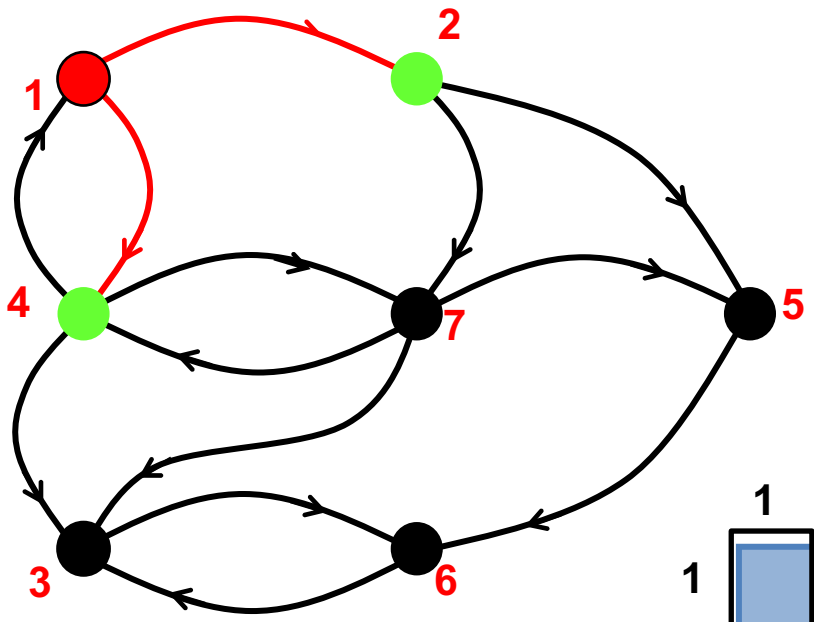
- Sparse graph representation: m+n
- Stack of visited vertices: n
- Distance array: n

Breadth-first search is a very important ***building block*** for other parallel graph algorithms such as (bipartite) matching, maximum flow, (strongly) connected components, betweenness centrality, etc.




# Breadth-first search using matrix algebra

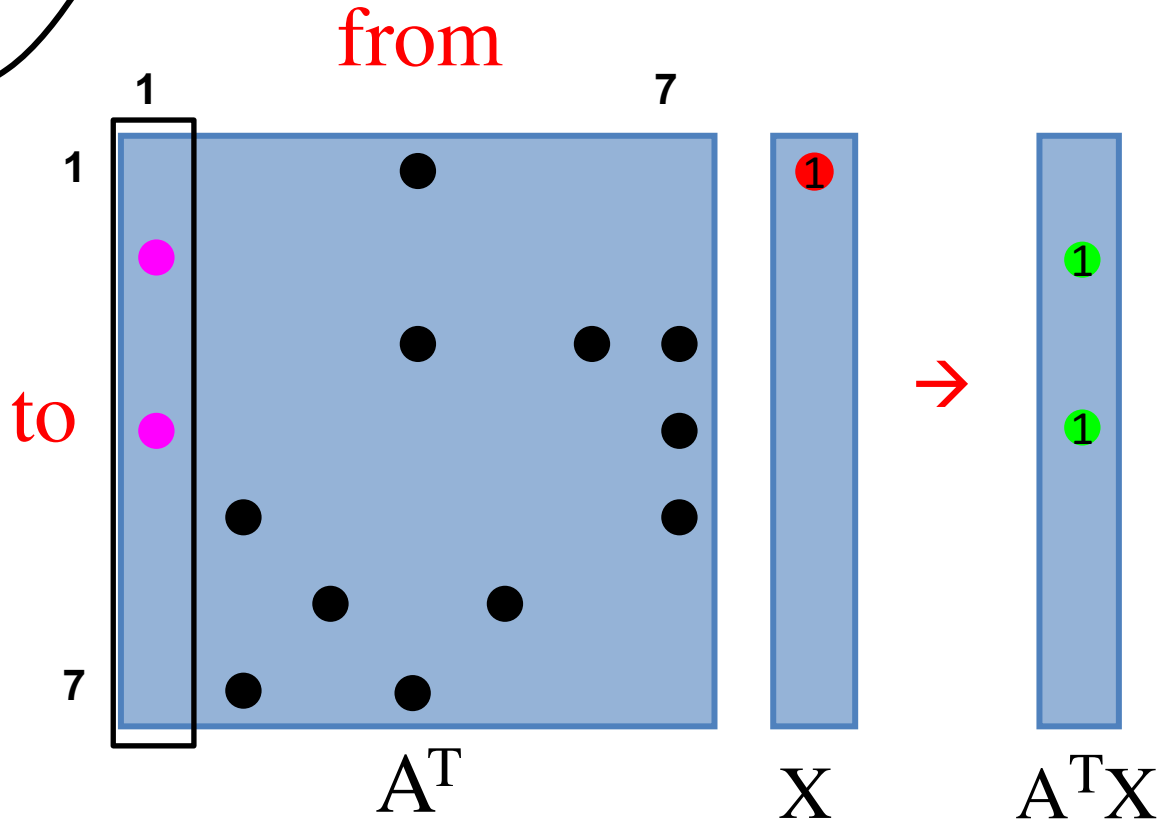




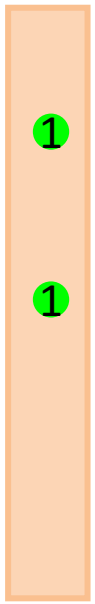

Replace scalar operations

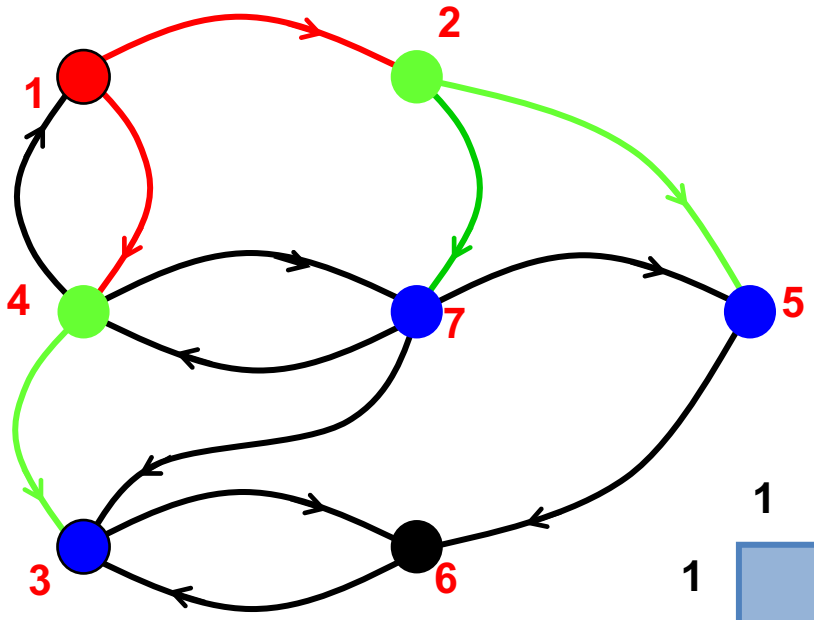
**Multiply ->** select

**Add ->** minimum



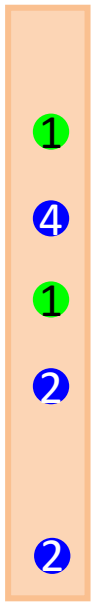

parents:





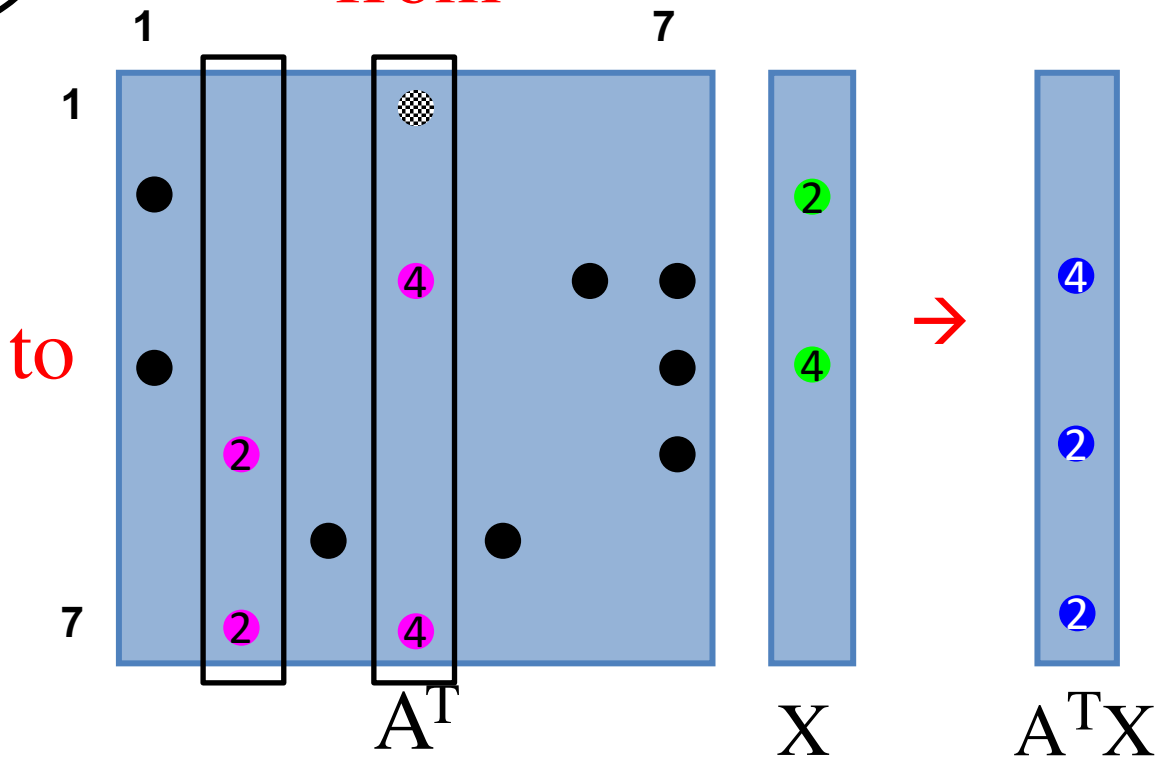

Select vertex with
minimum label as parent

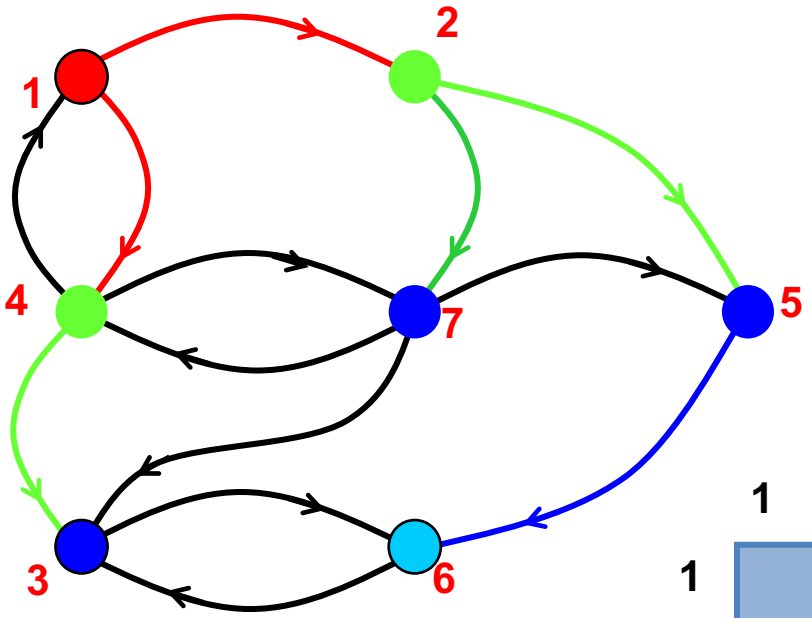
parents:



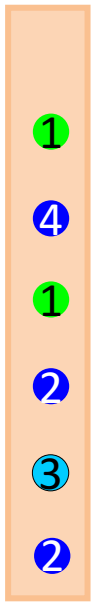

from

to






parents:

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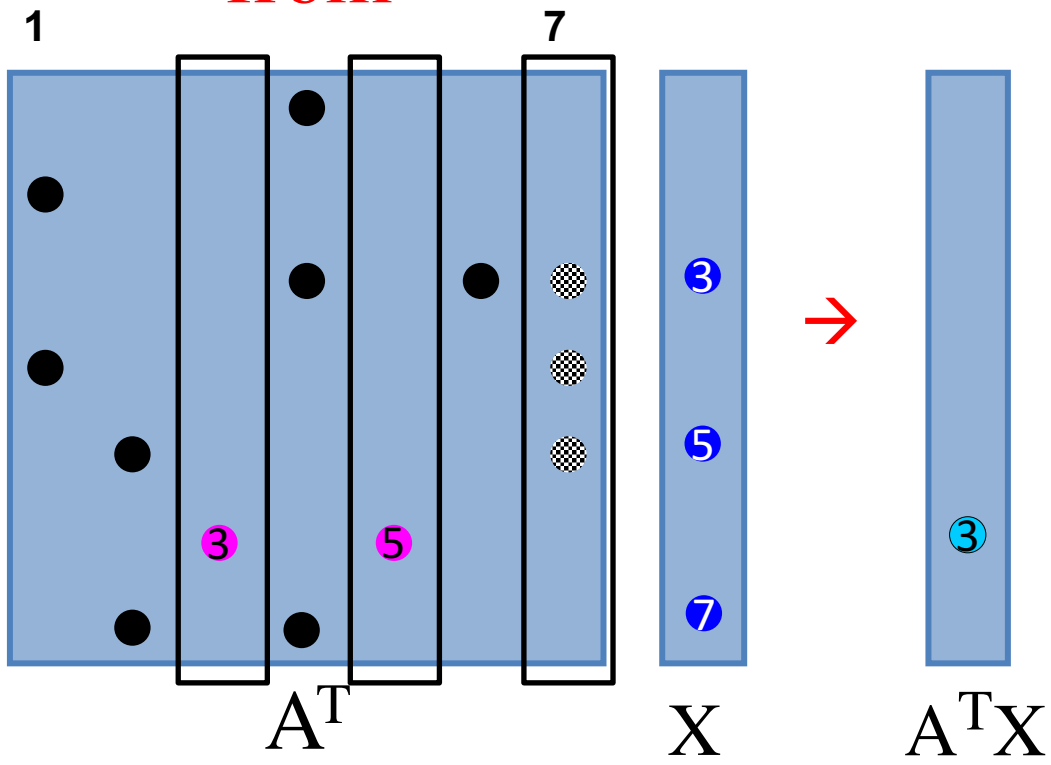

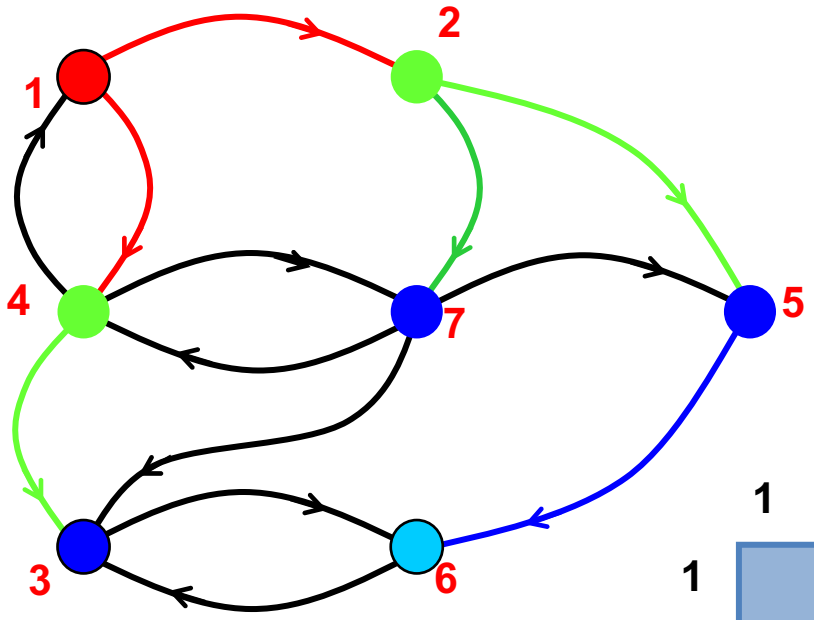


from

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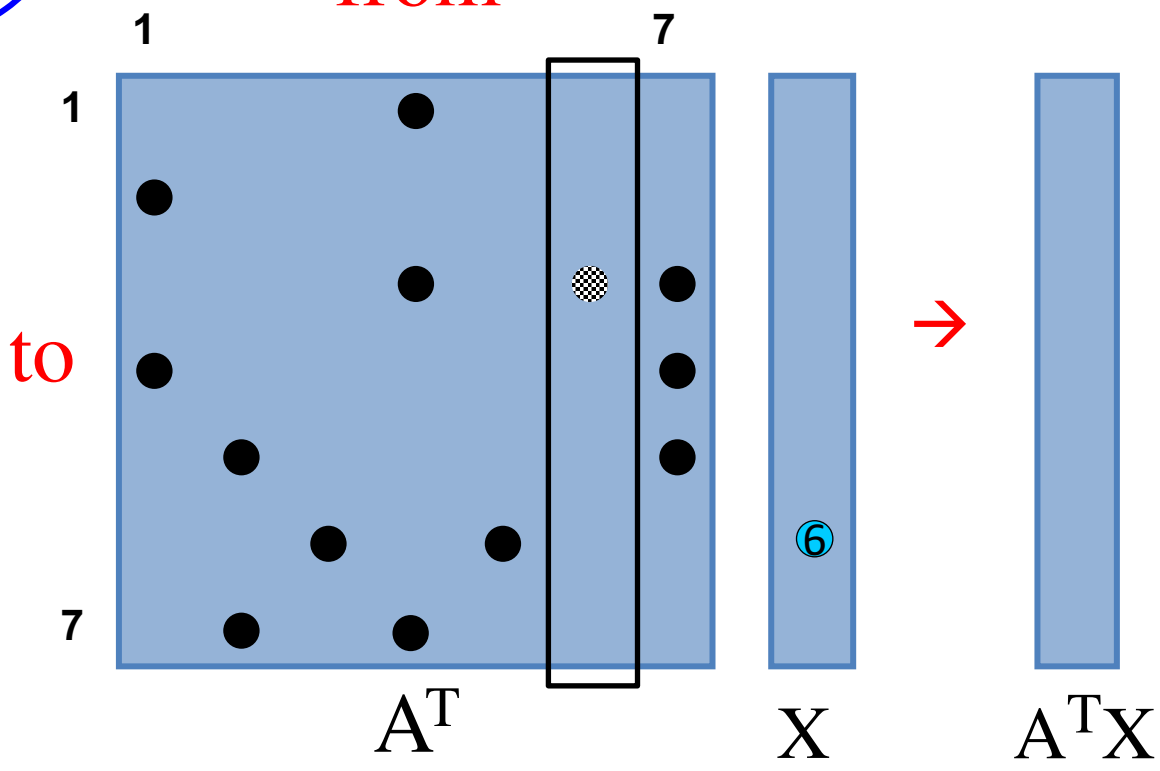
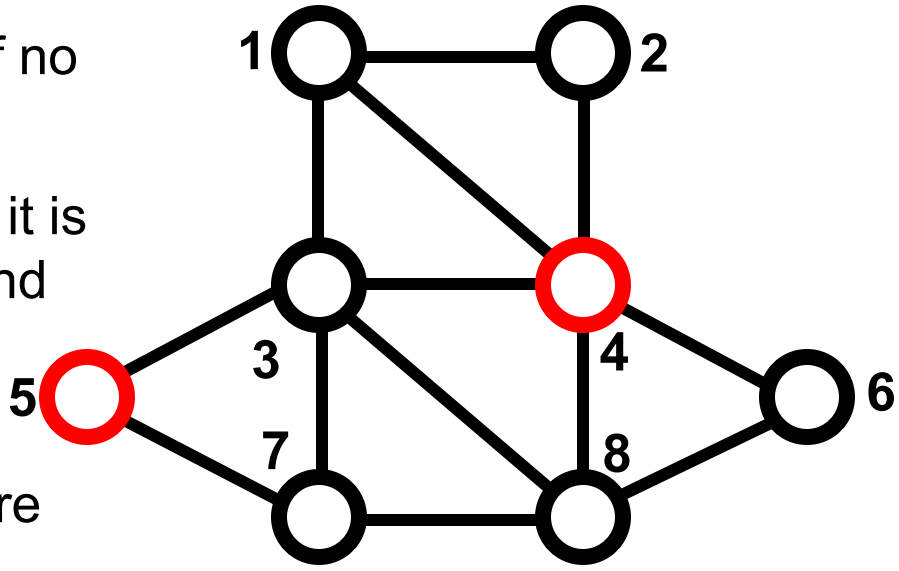


# Breadth-First Search in GraphBLAS

```
GrB_Vector q;                                   // vertices visited in each level
GrB_Vector_new(&q,GrB_BOOL,n);                  // Vector<bool> q(n) = false
GrB_Vector_setElement(q,(bool)true,s);          // q[s] = true, false everywhere else

GrB_Monoid Lor;                                 // Logical-or monoid
GrB_Monoid_new(&Lor,GrB_LOR,false);

GrB_Semiring Boolean;                           // Boolean semiring
GrB_Semiring_new(&Boolean,Lor,GrB_LAND);

GrB_Descriptor desc;                            // Descriptor for vxm
GrB_Descriptor_new(&desc);
GrB_Descriptor_set(desc,GrB_MASK,GrB_SCMP);     // invert the mask
GrB_Descriptor_set(desc,GrB_OUTP,GrB_REPLACE);// clear the output before assignment

GrB_UnaryOp apply_level;
GrB_UnaryOp_new(&apply_level,return_level,GrB_INT32,GrB_BOOL);

/*
 * BFS traversal and label the vertices.
 */
level = 0;
GrB_Index nvals;
do {
  ++level;                                      // next level (start with 1)
  GrB_apply(*v,GrB_NULL,GrB_PLUS_INT32,apply_level,q,GrB_NULL);  // v[q] = level
  GrB_vxm(q,*v,GrB_NULL,Boolean,q,A,desc);      // q[!v] = q ||.&& A ; finds all the
                                                // unvisited successors from current q
  GrB_Vector_nvals(&nvals, q);
} while (nvals);                                // if there is no successor in q, we are done.
```

# Maximal Independent Set

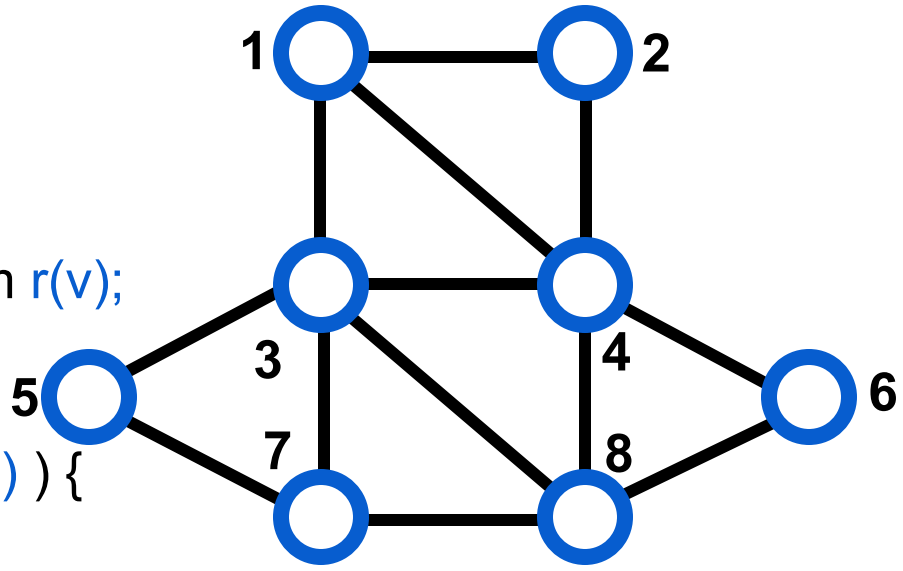
- Graph with vertices V = {1,2,…,n}
- A set S of vertices is independent if no two vertices in S are neighbors.
- An independent set S is maximal if it is impossible to add another vertex and stay independent
- An independent set S is maximum if no other independent set has more vertices
- Finding a *maximum* independent set is intractably difficult (NP-hard)
- Finding a *maximal* independent set is easy, at least on one processor.




The set of red vertices
S = {4, 5} is *independent*
and is *maximal*
but not *maximum*

# Parallel, Randomized MIS Algorithm

```
1.  S = empty set;  C = V;
2.  while  C  is not empty {
3.      label each v in C with a random r(v);
4.      for all v in C in parallel {
5.          if r(v) < min( r(neighbors of v) ) {
6.              move v from C to S;
7.              remove neighbors of v from C;
8.          }
9.      }
10. }
```




**S = { }**

**C = { 1, 2, 3, 4, 5, 6, 7, 8 }**

M. Luby. "A Simple Parallel Algorithm for the Maximal Independent Set Problem". *SIAM Journal on Computing* **15** (4): 1036–1053, 1986

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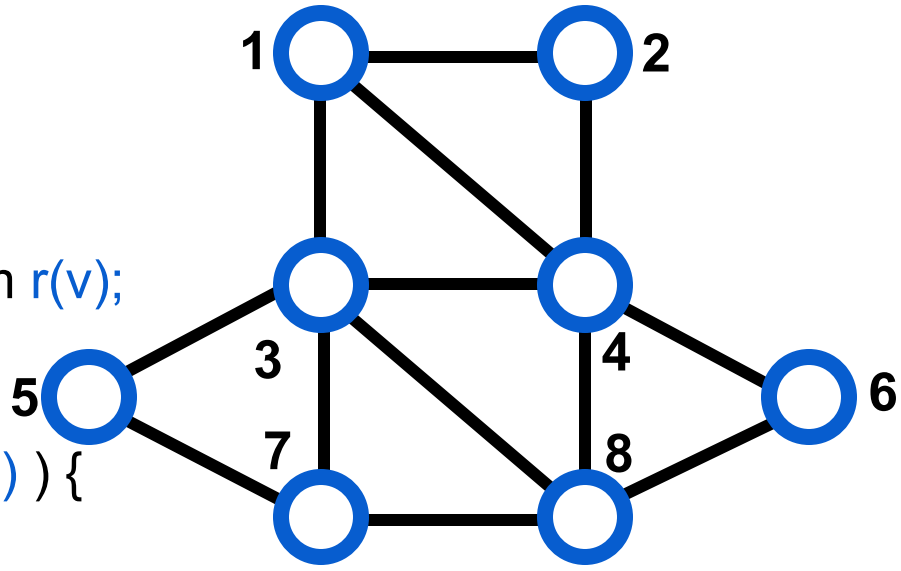


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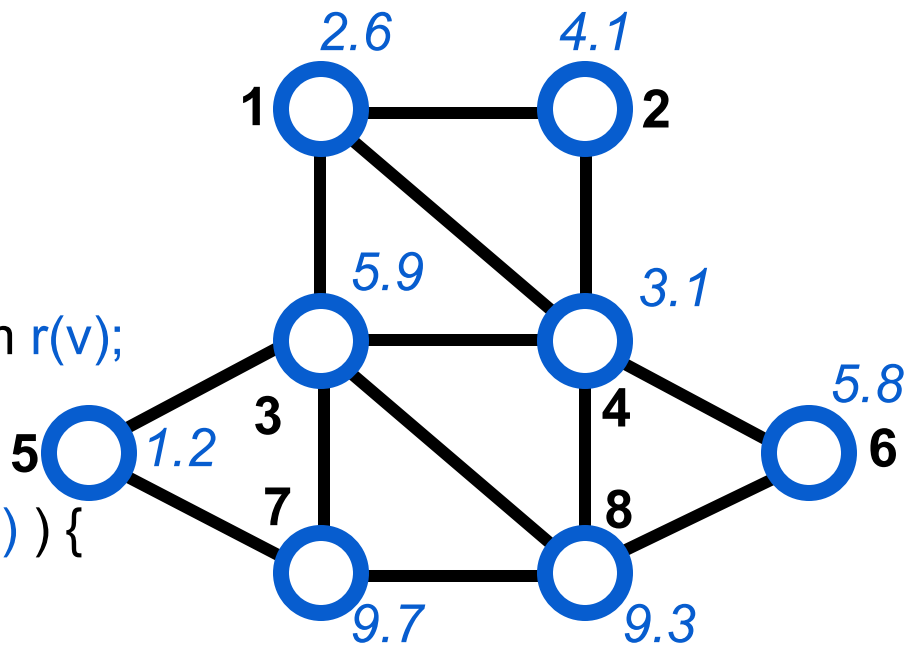


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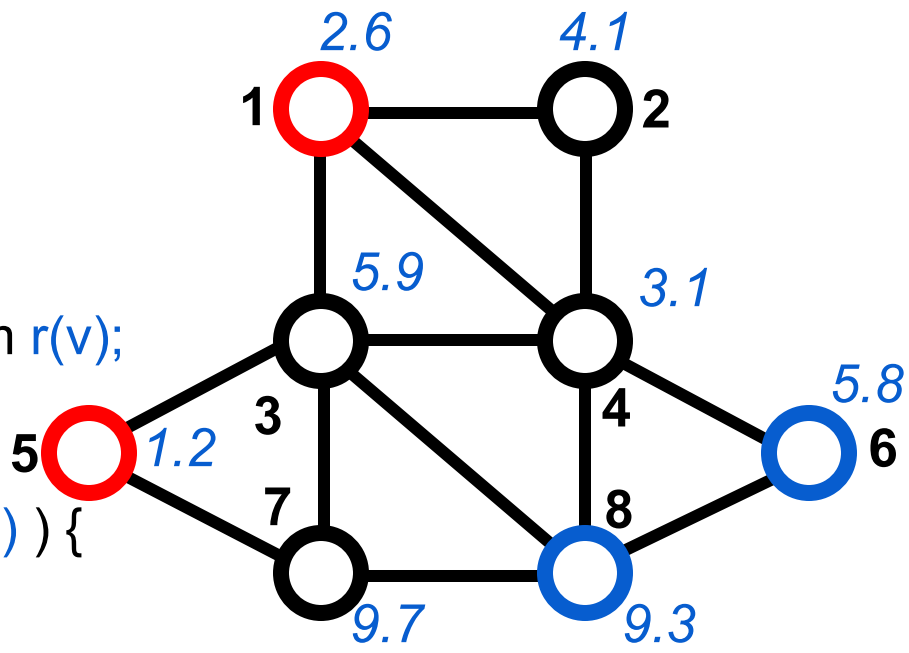


**S = { 1, 5 }**

**C = { 6, 8 }**

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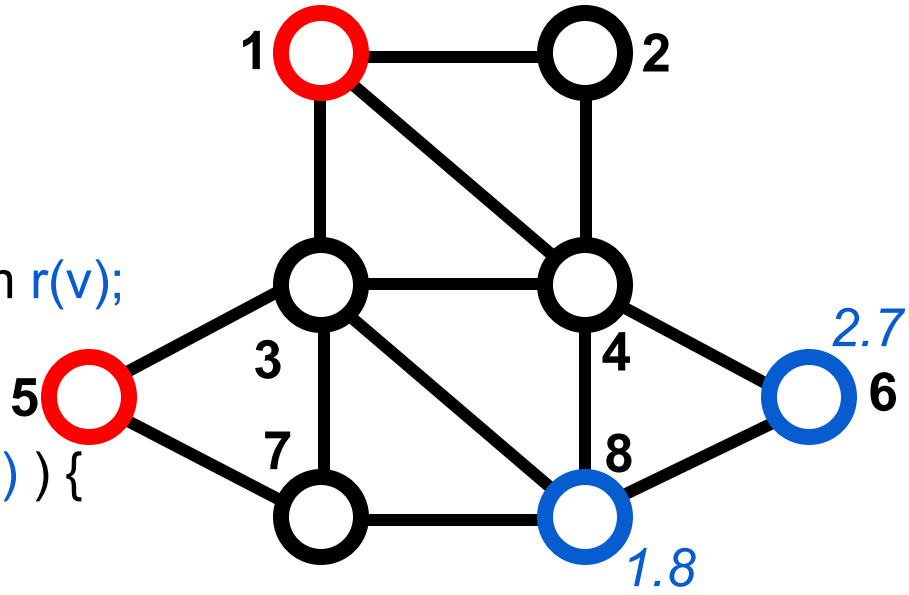


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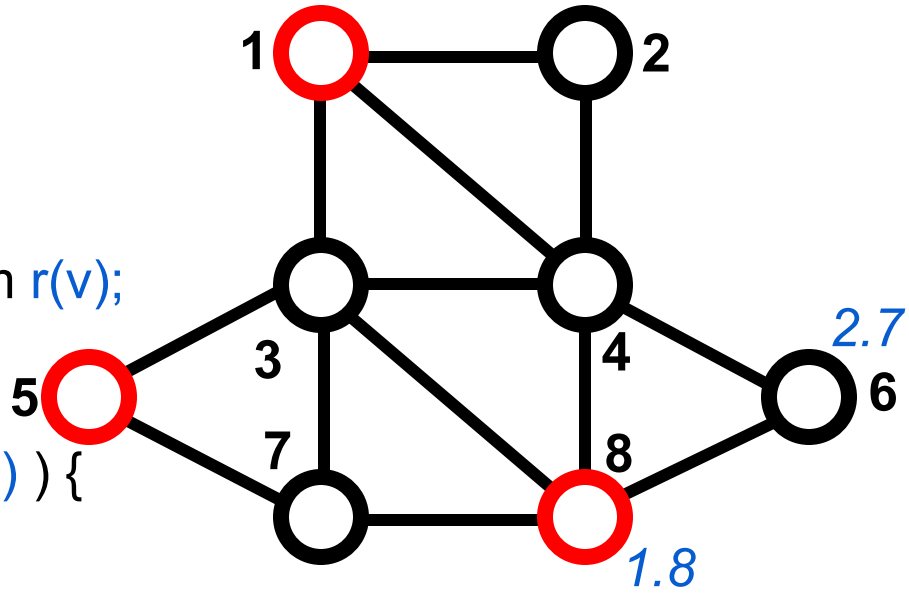


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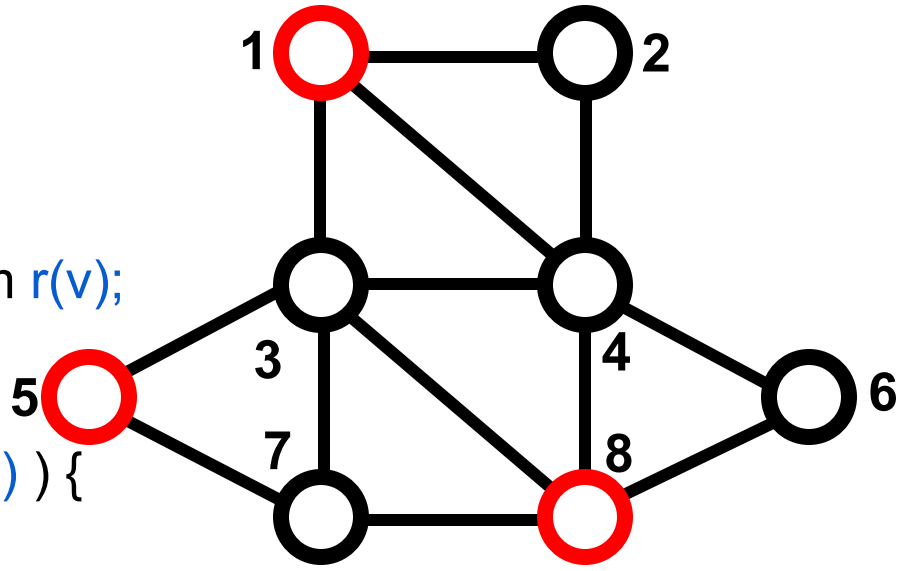



**Theorem: This algorithm "very probably" finishes within O(log n) rounds.**

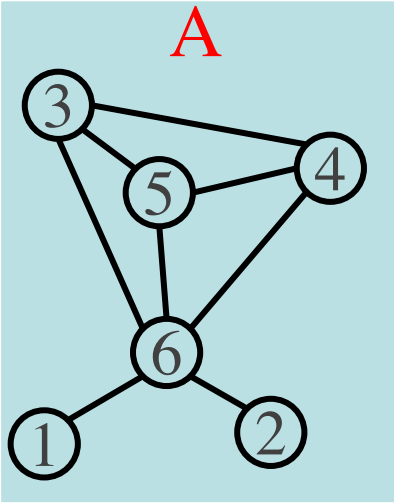
***work*** **~ O(n log n), but** ***span*** **~O(log n)**

# A Variant of Luby's Algorithm in GraphBLAS

```
// Iterate while there are candidates to check.
GrB_Index nvals;
GrB_Vector_nvals(&nvals, candidates);
while (nvals > 0) {
  // compute a random probability scaled by inverse of degree
  GrB_apply(prob,candidates,GrB_NULL,set_random,degrees,r_desc);

  // compute the max probability of all neighbors
  GrB_mxv(neighbor_max,candidates,GrB_NULL,maxSelect2nd,A,prob,r_desc);

  // select vertex if its probability is larger than all its active neighbors,
  // and apply a "masked no-op" to remove stored falses
  GrB_eWiseAdd(new_members,GrB_NULL,GrB_NULL,GrB_GT_FP64,prob,neighbor_max,GrB_NULL);
  GrB_apply(new_members,new_members,GrB_NULL,GrB_IDENTITY_BOOL,new_members,r_desc);

  // add new members to independent set.
  GrB_eWiseAdd(*iset,GrB_NULL,GrB_NULL,GrB_LOR,*iset,new_members,GrB_NULL);

  // remove new members from set of candidates c = c & !new
  GrB_eWiseMult(candidates,new_members,GrB_NULL,
                GrB_LAND,candidates,candidates,sr_desc);

  GrB_Vector_nvals(&nvals, candidates);
  if (nvals == 0) { break; }                    // early exit condition

  // Neighbors of new members can also be removed from candidates
  GrB_mxv(new_neighbors,candidates,GrB_NULL,Boolean,A,new_members,GrB_NULL);
  GrB_eWiseMult(candidates,new_neighbors,GrB_NULL,
                GrB_LAND,candidates,candidates,sr_desc);

  GrB_Vector_nvals(&nvals, candidates);
}
```

http://graphblas.org

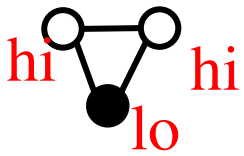
# Counting triangles



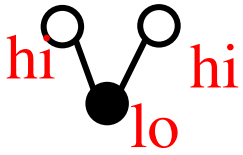

Clustering coefficient:

- Pr (wedge i-j-k makes a triangle with edge i-k)
- 3 * # triangles / # wedges
- 3 * **4** / **19** = **0.63** in example
- may want to compute for each vertex j

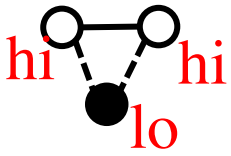
## Cohen's algorithm to count triangles:




- Count triangles by lowest-degree vertex.

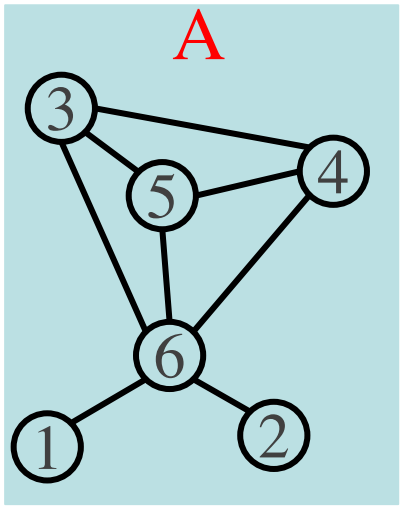



- Enumerate "low-hinged" wedges.




- Keep wedges that close.

# Counting triangles

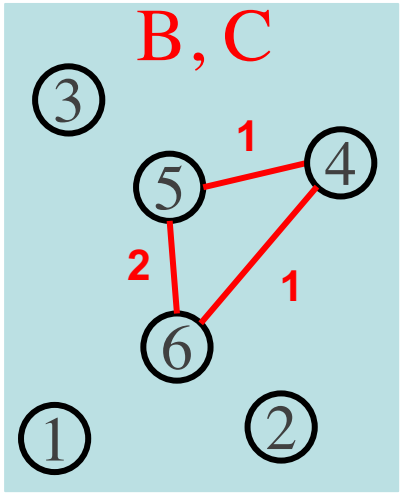


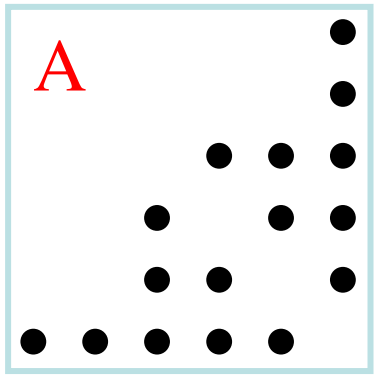

$A = L + U$ (hi->lo + lo->hi)

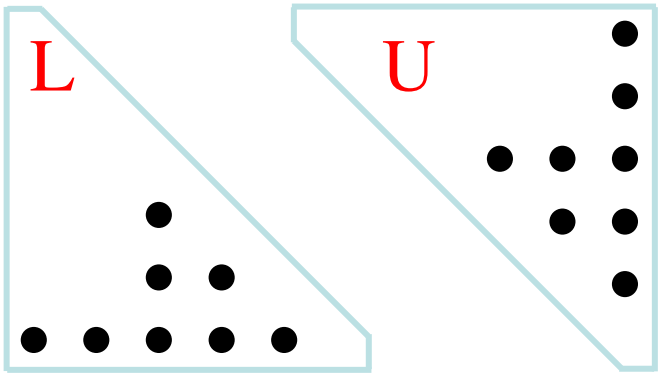
$L \times U = B$ (wedge, low hinge)

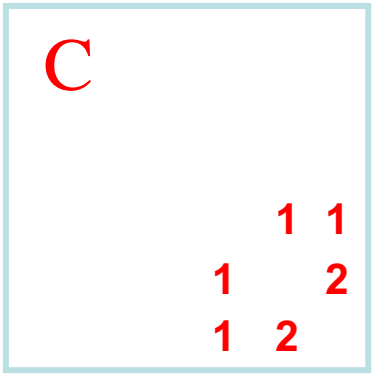
$A \wedge B = C$ (closed wedge)

$sum(C)/2 =$ **4 triangles**



# Triangle Counting in GraphBLAS

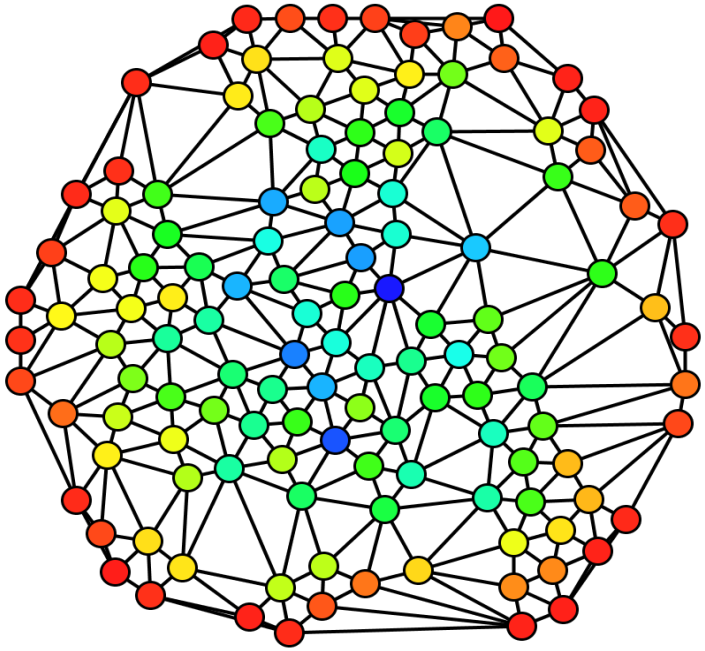
```
/*
 * Given, L, the lower triangular portion of n x n adjacency matrix A (of and
 * undirected graph), computes the number of triangles in the graph.
 */
uint64_t triangle_count(GrB_Matrix L)                  // L: NxN, lower-triangular, bool
{
  GrB_Index n;
  GrB_Matrix_nrows(&n, L);                             // n = # of vertices

  GrB_Matrix C;
  GrB_Matrix_new(&C, GrB_UINT64, n, n);

  GrB_Monoid UInt64Plus;                               // integer plus monoid
  GrB_Monoid_new(&UInt64Plus,GrB_PLUS_UINT64,0ul);

  GrB_Semiring UInt64Arithmetic;                       // integer arithmetic semiring
  GrB_Semiring_new(&UInt64Arithmetic,UInt64Plus,GrB_TIMES_UINT64);

  GrB_Descriptor desc_tb;                              // Descriptor for mxm
  GrB_Descriptor_new(&desc_tb);
  GrB_Descriptor_set(desc_tb,GrB_INP1,GrB_TRAN); // transpose the second matrix

  GrB_mxm(C, L, GrB_NULL, UInt64Arithmetic, L, L, desc_tb); // C<L> = L *.+ L'

  uint64_t count;
  GrB_reduce(&count, GrB_NULL, UInt64Plus, C, GrB_NULL);     // 1-norm of C

  GrB_free(&C);                          // C matrix no longer needed
  GrB_free(&UInt64Arithmetic);           // Semiring no longer needed
  GrB_free(&UInt64Plus);                 // Monoid no longer needed
  GrB_free(&desc_tb);                    // descriptor no longer needed

  return count;
}
```

http://graphblas.org

# Conclusions

- GraphBLAS enables one to efficiently cast graph algorithms and machine learning methods into the languages of sparse matrices
- While elegant and efficient for problems that fit into the linear algebra framework, it is admittedly not fully universal.
- The standard definition by the C API group and a compliant implementation by Tim Davis available at http://graphblas.org
- More parallel implementations in the works. Currently one can use approximate GraphBLAS implementations from Combinatorial BLAS, Kokkos, and Cyclops Tensor Framework.

# Betweenness Centrality



**Definition:**

**$C_B(v)$:** Among all the shortest paths, what fraction of them pass through the node of interest?

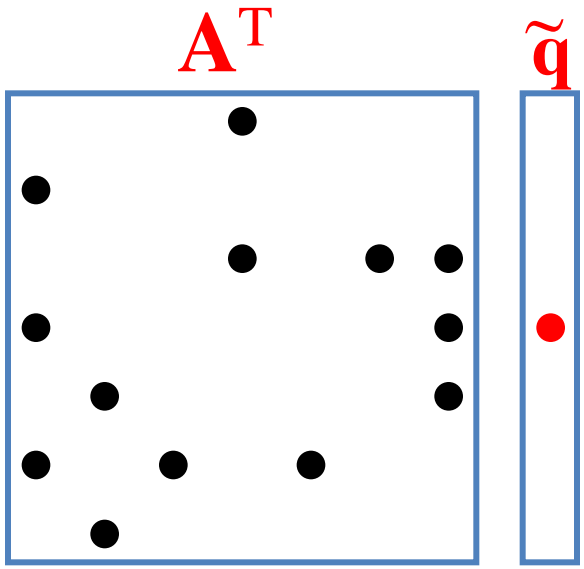
$$BC(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

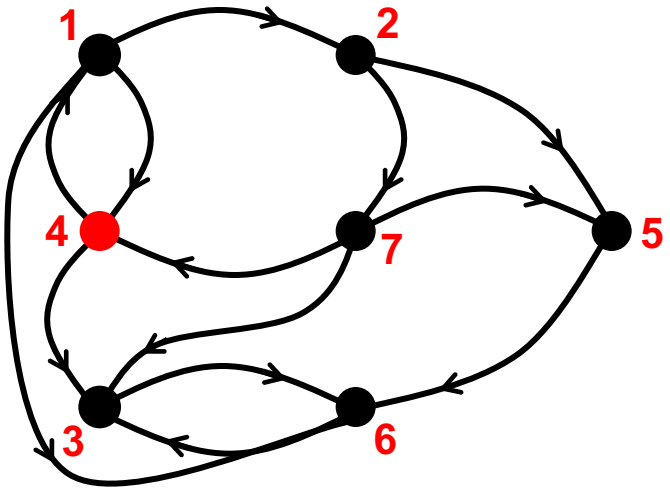
$\sigma_{st}$ is the number of shortest paths between vertices s and t

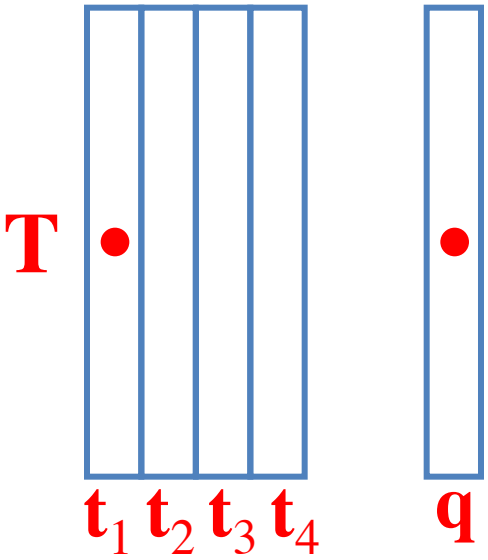
$\sigma_{st}$ (v) is the number of such paths that pass through vertex v

- APSP is wasteful for sparse graphs
- Brandes' algorithm is O(mn) for unweighted graphs

# Betweenness Centrality: Data Structures

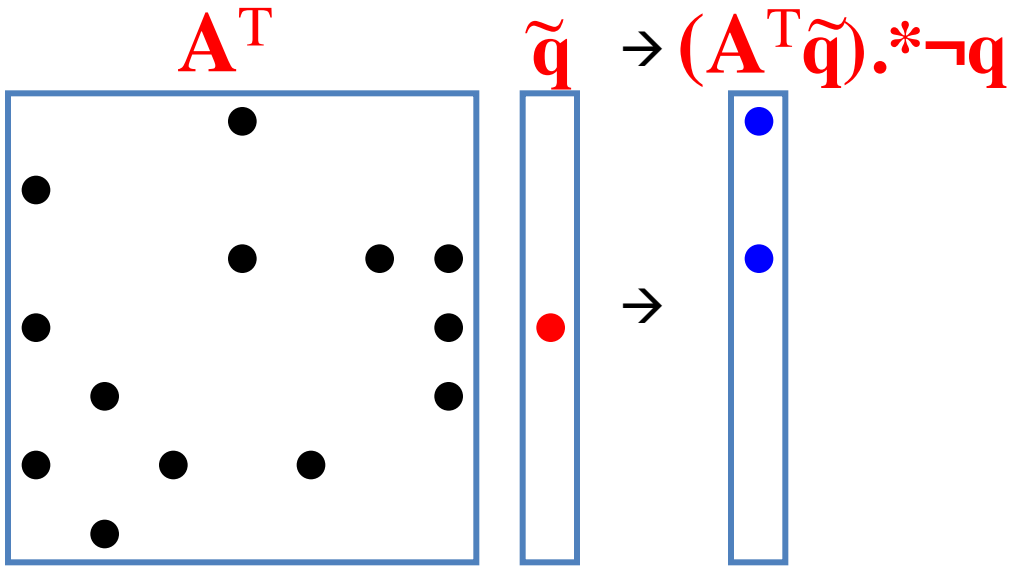


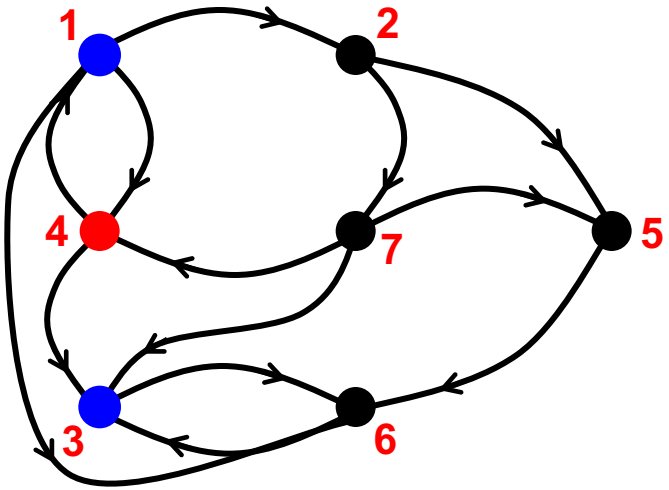



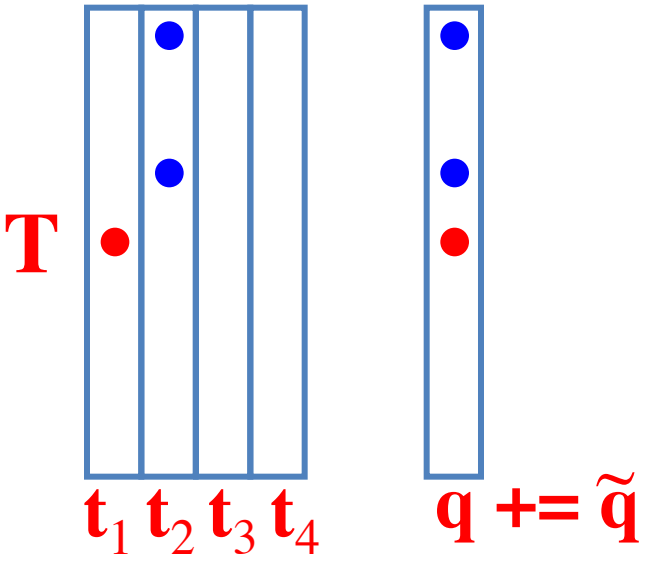




- Pick a starting vertex (4)
- Initialize vectors: q, $\tilde{q}$, and $t_d$

# Betweenness Centrality: Get Neighbors

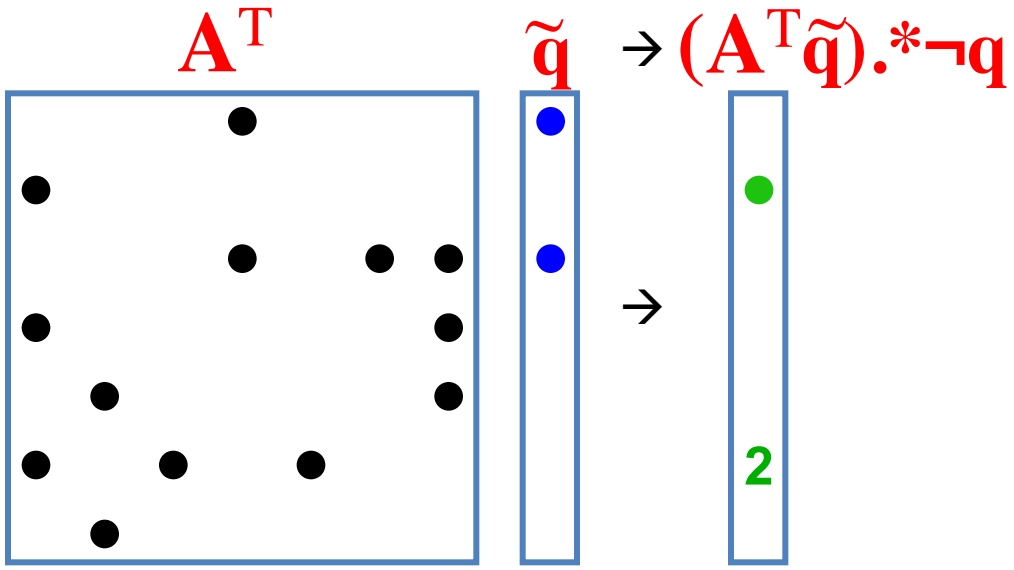


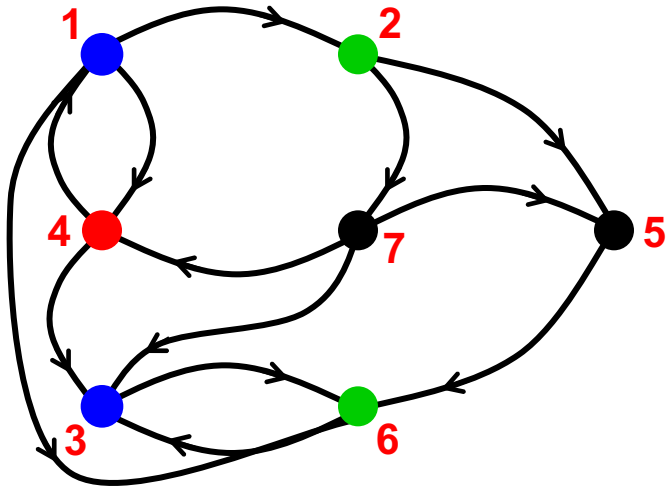



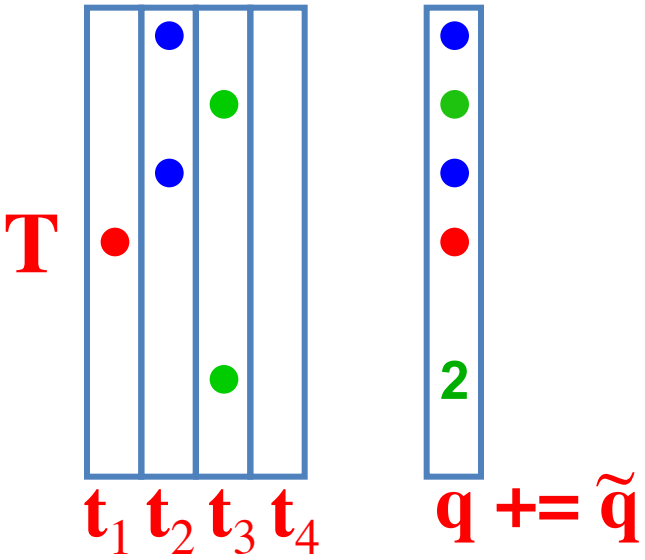




- Get 2nd neighbors from starting vertex: $A^T \tilde{q}$
- Eliminate existing vertices: $.*\neg q$
- Tally: $q \mathrel{+}= \tilde{q}$
- Update table: $t_2 = \tilde{q}$

# Betweenness Centrality: Get Neighbors

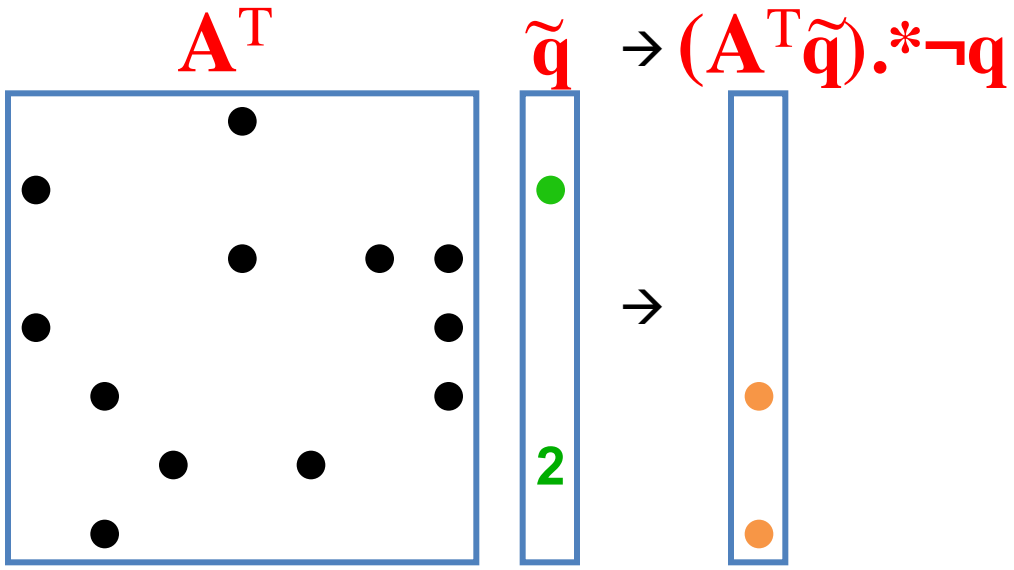


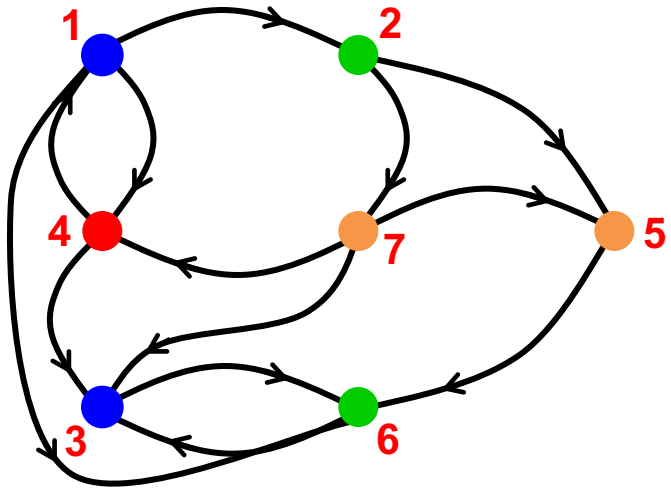



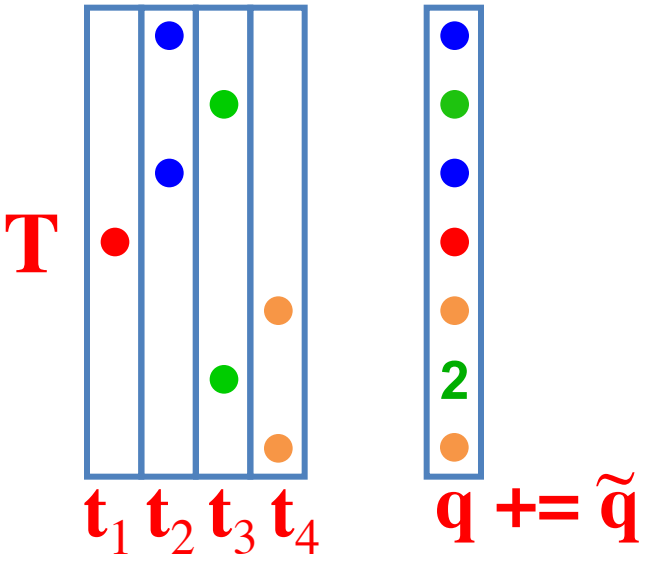




- Get 3rd neighbors from starting vertex: $A^T \tilde{q}$ ; sum paths to vertex
- Eliminate existing vertices: .*¬q
- Tally: q+=q̃
- Update table: $t_2 = \tilde{q}$

# Betweenness Centrality: Get Neighbors

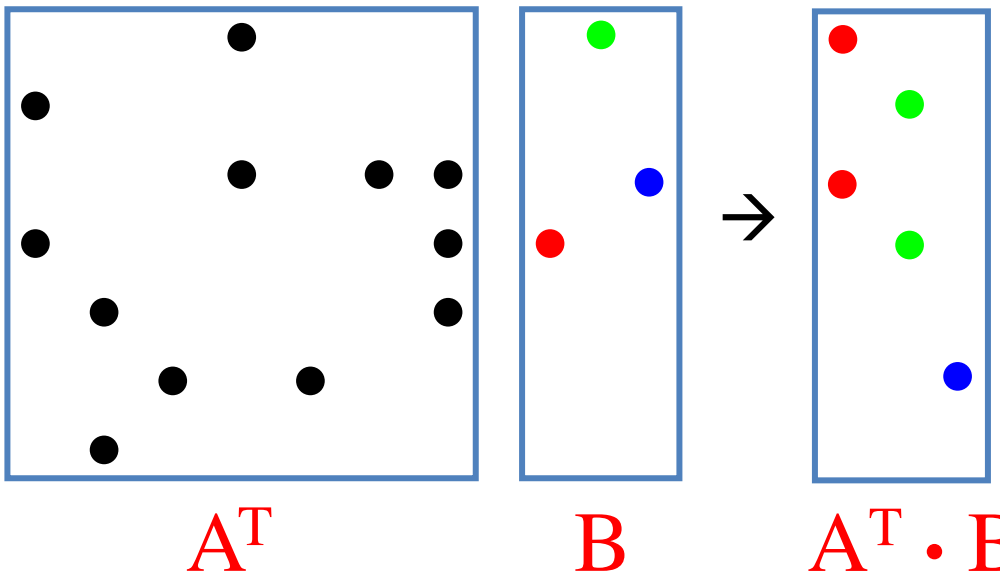


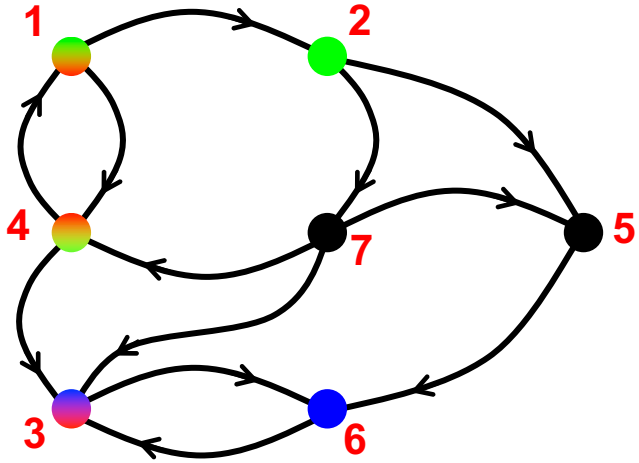




- Get 4th neighbors from starting vertex: $A^T\tilde{q}$
- Eliminate existing vertices: $.*\neg q$
- Tally: $q+=\tilde{q}$
- Update table: $t_2 = \tilde{q}$

# Driver: Multiple-source breadth-first search




- Sparse array representation => space efficient
- Sparse matrix-matrix multiplication => work efficient
- **Three possible levels of parallelism: searches, vertices, edges**
- Highly-parallel implementation for Betweenness Centrality*

*: A measure of influence in graphs, based on shortest paths

# Forward sweep of BC in GraphBLAS C API

```
#include "GraphBLAS.h"

GrB_Info BC_update(GrB_Vector *delta, GrB_Matrix A, GrB_Index *s, GrB_Index nsver)
{
  GrB_Index n;
  GrB_Matrix_nrows(&n, A);                              // n = # of vertices in graph
  GrB_Vector_new(delta,GrB_FP32,n);                     // Vector<float> delta(n)
  GrB_Monoid Int32Add;                                  // Monoid <int32_t,+,0>
  GrB_Monoid_new(&Int32Add,GrB_INT32,GrB_PLUS_INT32,0);
  GrB_Semiring Int32AddMul;               // Semiring <int32_t,int32_t,int32_t,+,*,0>
  GrB_Semiring_new(&Int32AddMul,Int32Add,GrB_TIMES_INT32);

  GrB_Descriptor desc_tsr;                              // Descriptor for BFS phase mxm

  GrB_Descriptor_new(&desc_tsr);
  GrB_Descriptor_set(desc_tsr,GrB_INP0,GrB_TRAN);     // transpose of the adjacency matrix
  GrB_Descriptor_set(desc_tsr,GrB_MASK,GrB_SCMP);     // structural complement of the mask
  GrB_Descriptor_set(desc_tsr,GrB_OUTP,GrB_REPLACE);  // clear output before result is stored

  // index and value arrays needed to build numsp
  GrB_Index *i_nsver = malloc(sizeof(GrB_Index)*nsver);
  int32_t   *ones    = malloc(sizeof(int32_t)*nsver);
  for(int i=0; i<nsver; ++i) {
    i_nsver[i] = i;
    ones[i] = 1;
  }
…
```

# Forward sweep of BC in GraphBLAS C API

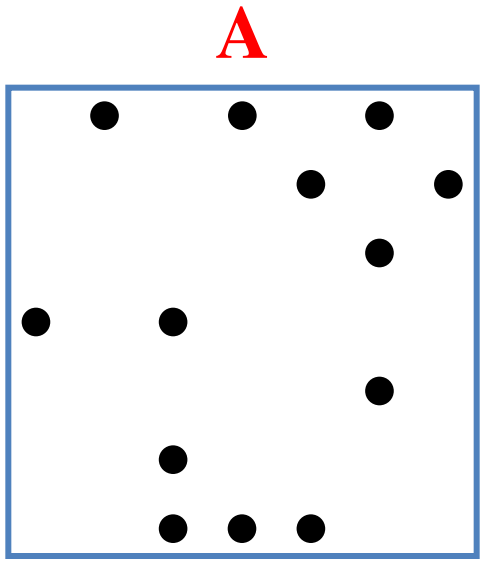
```
…
  GrB_Matrix numsp;  // Its nonzero structure holds all vertices that have been discovered
  GrB_Matrix_new( , GrB_INT32, n, nsver); // also stores # of shortest paths so far

  GrB_Matrix_build( ,GrB_NULL,GrB_NULL,s,i_nsver,ones,nsver,GrB_PLUS_INT32,GrB_NULL);
  free(i_nsver); free(ones);

  GrB_Matrix frontier;     // Holds the current frontier where values are path counts.
  GrB_Matrix_new(&frontier, GrB_INT32, n, nsver); // Initialized: neighbors of each source
  GrB_extract(&frontier,numsp,GrB_NULL,A,GrB_ALL,n,s,nsver,desc_tsr);

  // The memory for an entry in sigmas is only allocated within the do-while loop if needed
  GrB_Matrix *sigmas = malloc(sizeof(GrB_Matrix)*n);     // n is an upper bound on diameter
  int32_t d = 0;                                         // BFS level number
  int32_t nvals = 0;                                     // nvals == 0 when BFS phase is complete
  do {  // ---------------------- The BFS phase (forward sweep) --------------------------
    GrB_Matrix_new(&(sigmas[d]), GrB_BOOL, n, nsver);
    // sigmas[d](:,s) = d^th level frontier from source vertex s

    GrB_apply(&(sigmas[d]),GrB_NULL,GrB_NULL,GrB_IDENTITY_BOOL,frontier,GrB_NULL);
    GrB_eWiseAdd( ,GrB_NULL,GrB_NULL,Int32Add,numsp,frontier,GrB_NULL);
    // numsp += frontier (accum path counts)

    GrB_mxm(&frontier,numsp,GrB_NULL,Int32AddMul,A,frontier,desc_tsr);
    // f<!numsp> = A' +.* f (update frontier)
    GrB_Matrix_nvals(&nvals,frontier)
    d++;
  } while (nvals);
…
```

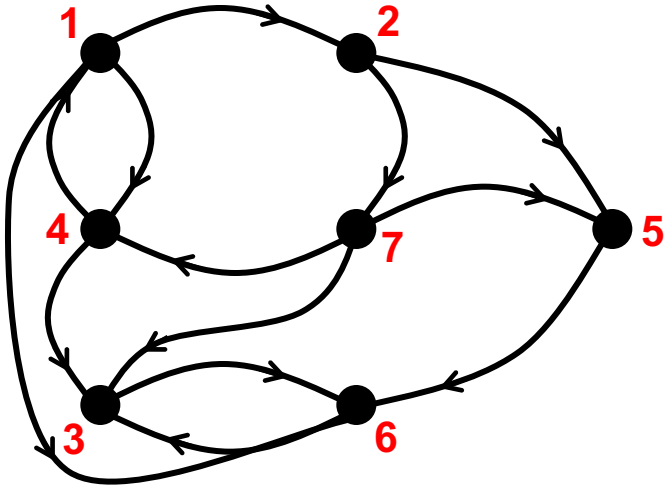
# Forward sweep of BC in GraphBLAS C API

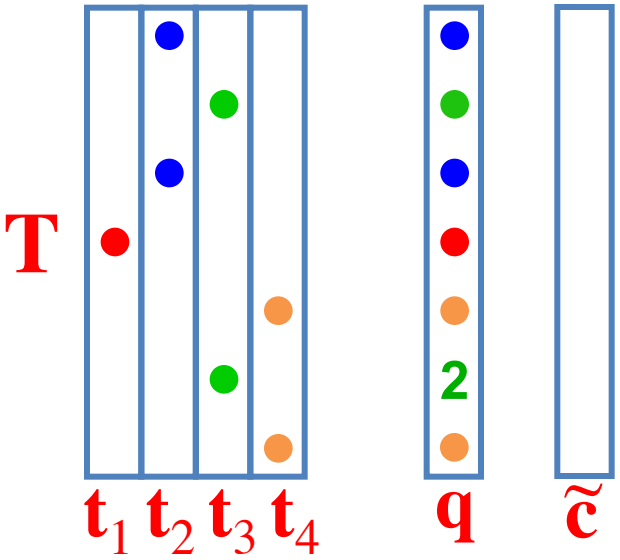
```
...
  GrB_Matrix nu
  GrB_Matrix_ne

  GrB_Matrix_bu
  free(i_nsver)

  GrB_Matrix fr
  GrB_Matrix_ne
  GrB_extract(&

  // The memory
  GrB_Matrix *s
  int32_t d = 0
  int32_t nvals
  do {  // ----
    GrB_Matrix_
    // sigmas[d

    GrB_apply(&
    GrB_eWiseAd
    // numsp += frontier (accum path counts)

    GrB_mxm(&frontier,numsp,GrB_NULL,Int32AddMul,A,frontier,desc_tsr);
    // f<!numsp> = A' +.* f (update frontier)
    GrB_Matrix_nvals(&nvals,frontier)
    d++;
  } while (nvals);
...
```

- The GrB_mxm call forms the next frontier in one step by both expanding the current frontier (i.e., discovering the 1-hop neighbors of the set of vertices in the current frontier) and pruning the vertices that have already been discovered.
- The former is achieved by setting the descriptor, desc_tsr, to use the transpose of the adjacency matrix. The latter is achieved by setting the descriptor to use the structural complement of the mask and by passing the numsp matrix as the mask parameter.
- The implicit cast of numsp to Boolean allows GrB_mxm to interpret numsp as the set of previously discovered vertices.
- Note that the descriptor is also set to GrB_REPLACE to ensure that the frontier is overwritten with new values.

# Betweenness Centrality: Roll back & Tally

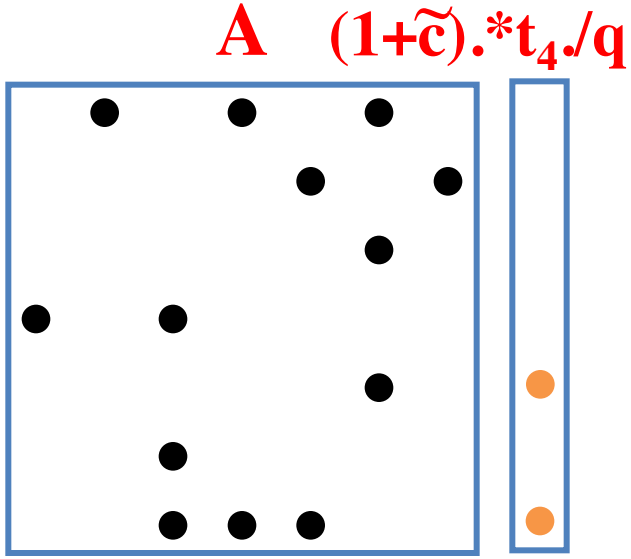


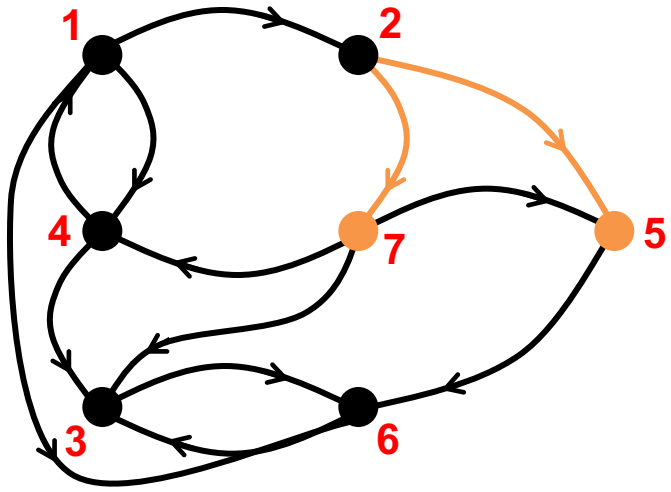



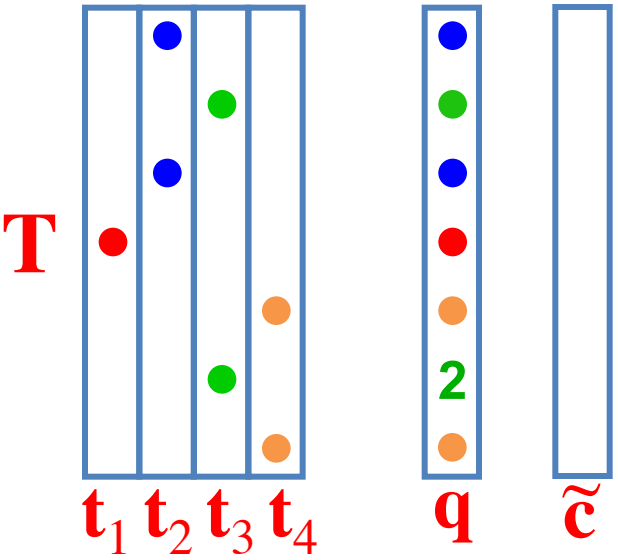




- Initialize the centrality update: $\tilde{c}$
- Will hold the contributions of these shortest paths to each vertexes betweenness centrality

# Betweenness Centrality: Roll back & Tally

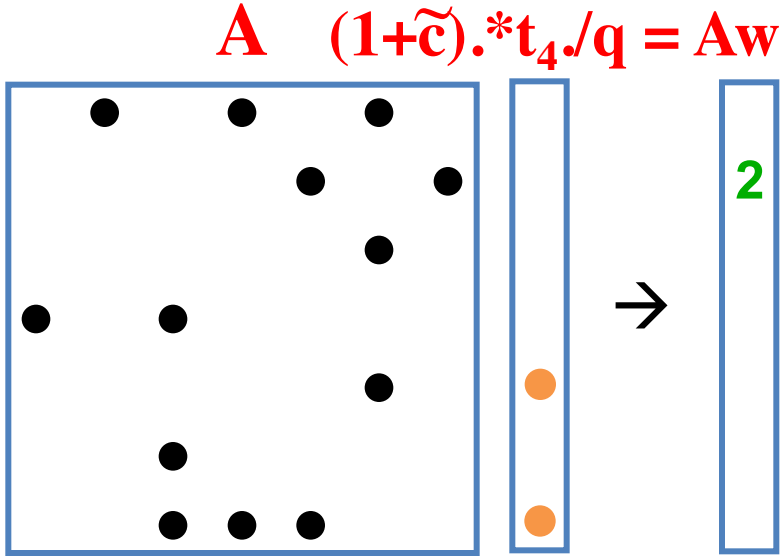


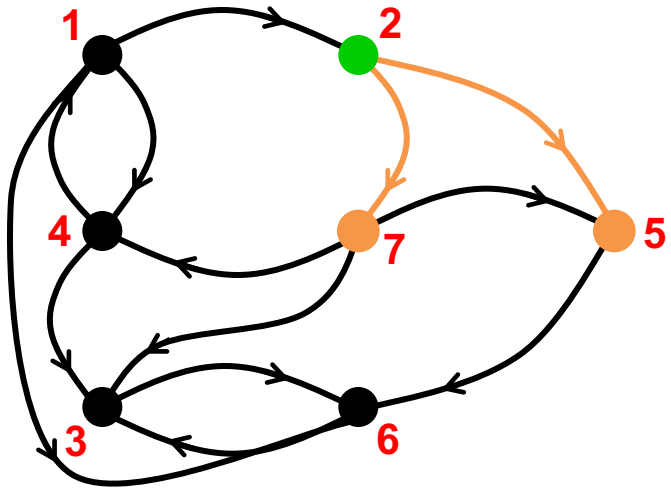



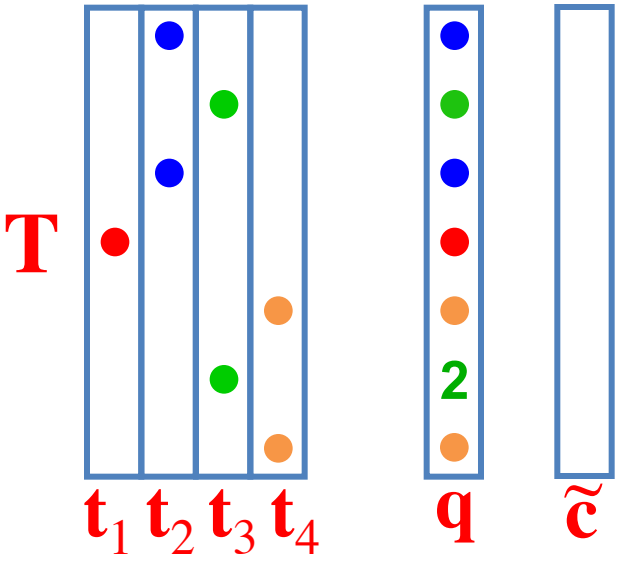




- Select 4th neighbors, divide by number of paths to these nodes:
  $(1+c).*t_4./q = w$

# Betweenness Centrality: Roll back & Tally

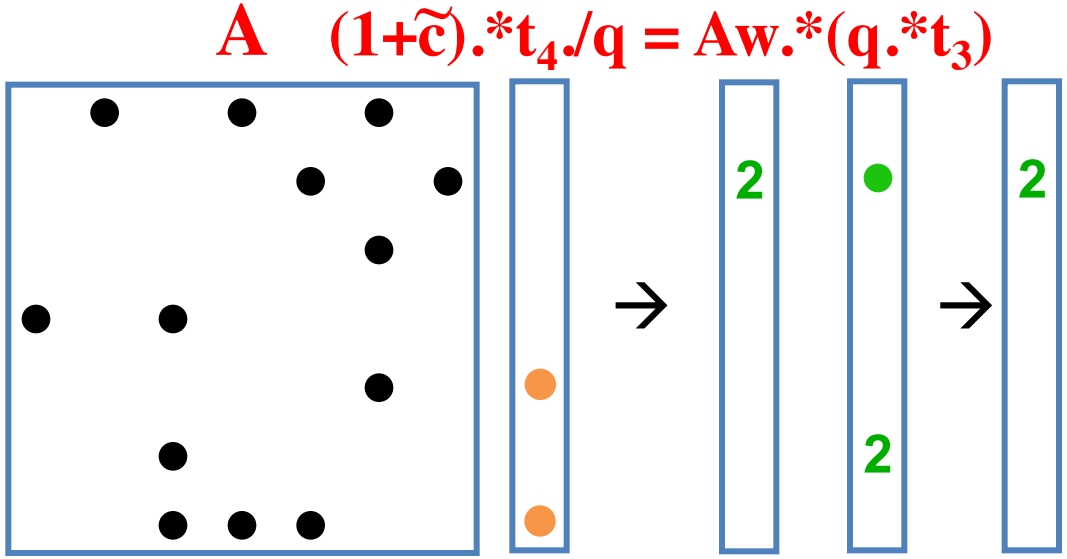


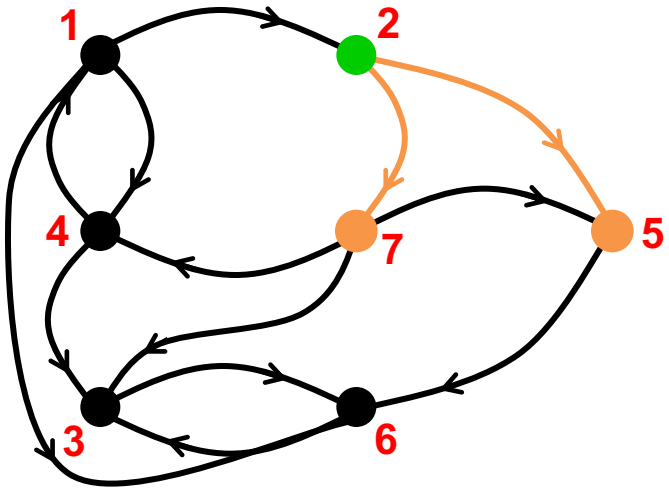



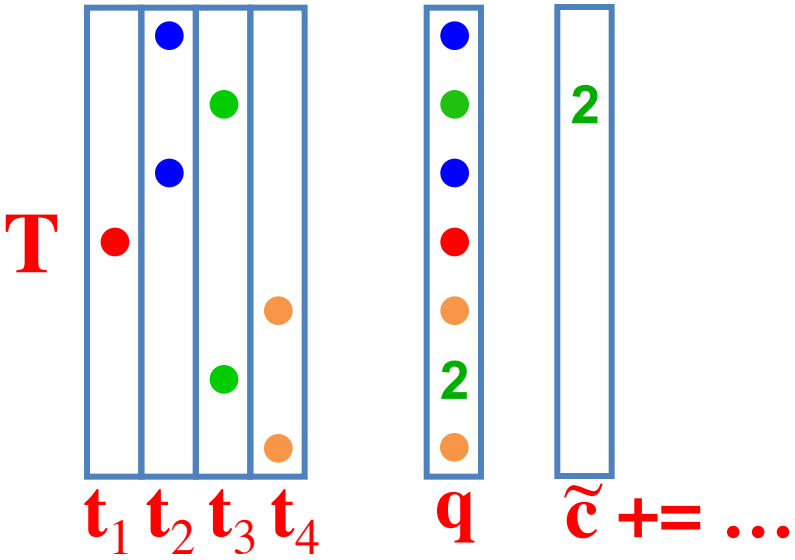




- Select 4th neighbors, divide by number of paths to these nodes: $(1+c).*t_4./q = w$
- Find 3rd neighbors: $Aw$

# Betweenness Centrality: Roll back & Tally

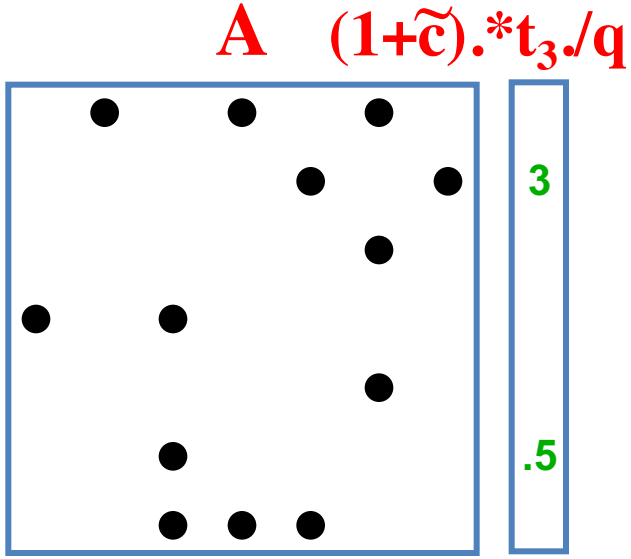


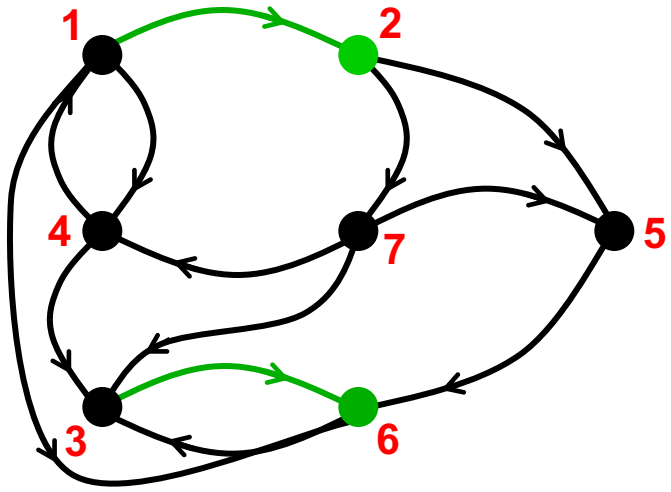



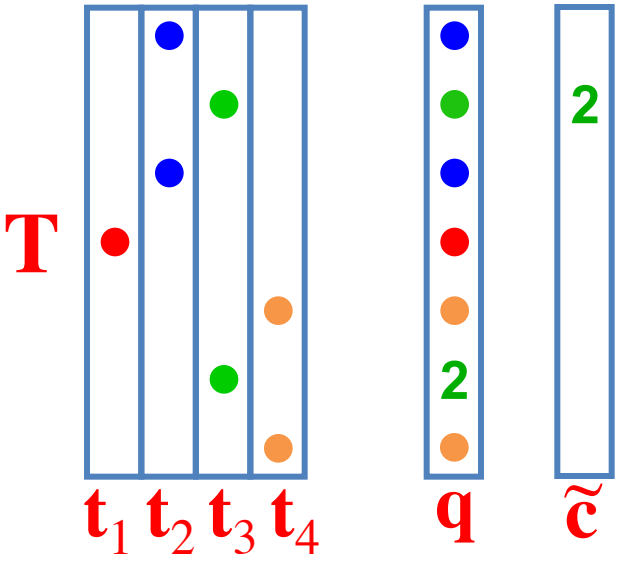




- Select 4th neighbors, divide by number of paths to these nodes: $(1+c).*t_4./q = w$
- Find 3rd neighbors: Aw
- Multiply by paths into 3rd neighbors and tally: $c+=Aw.*(q.*t_3)$

# Betweenness Centrality: Roll back & Tally




- Select 3rd neighbors, divide by number of paths to these nodes:
  $(1+c).*t_3./q = w$

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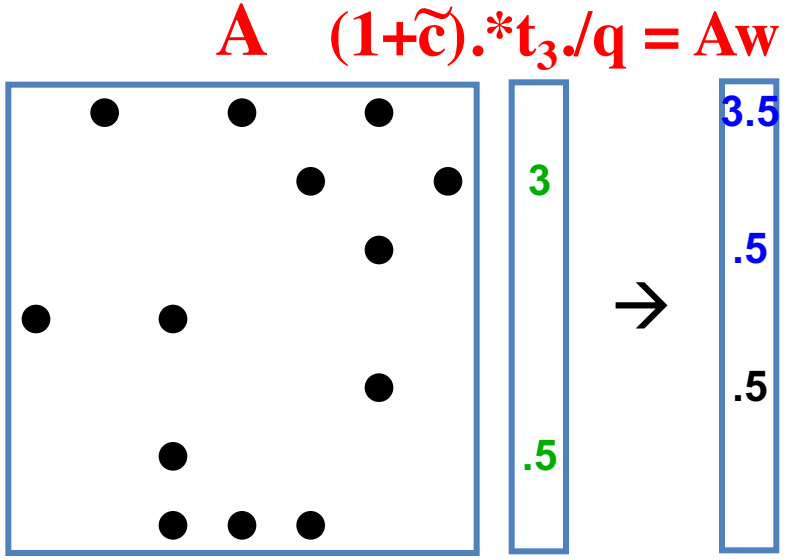


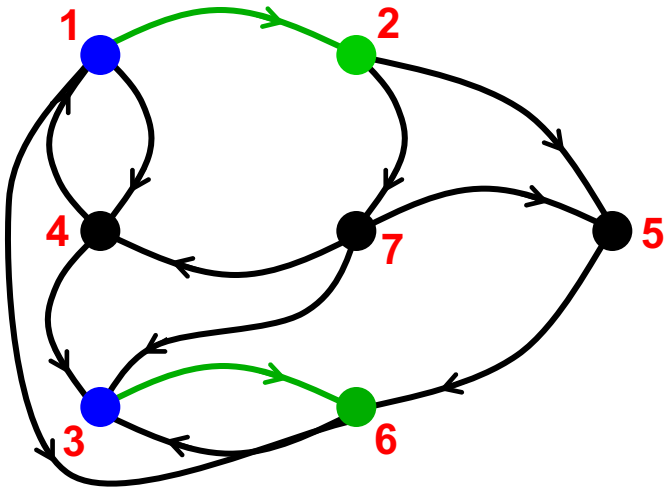


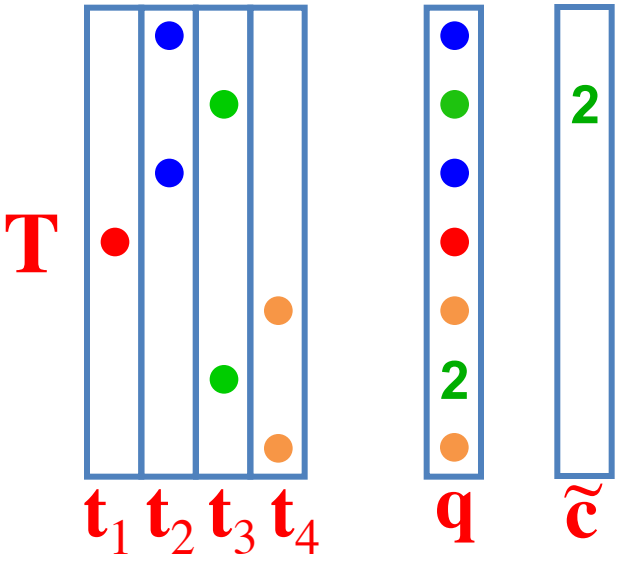


- Select 3rd neighbors, divide by number of paths to these nodes: $(1+c).*t_3./q = w$
- Find 2nd neighbors: Aw

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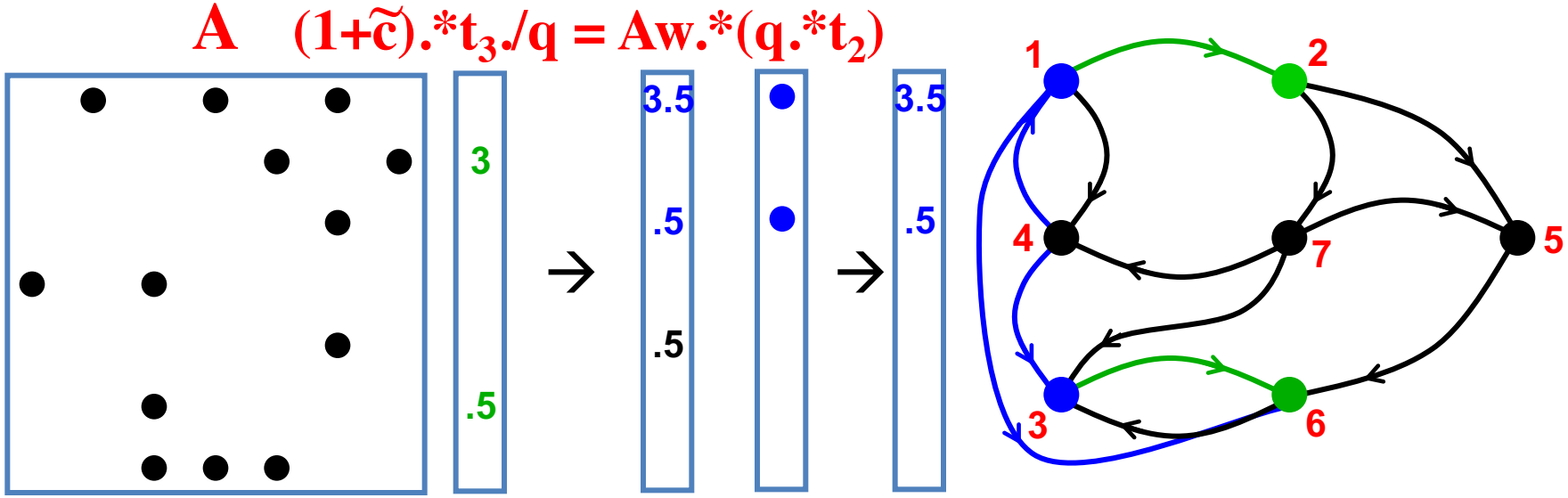


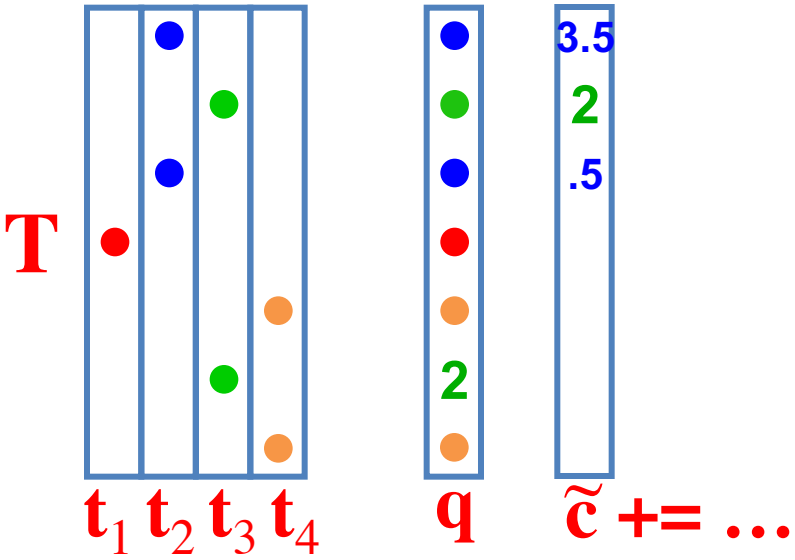


- Select 3rd neighbors, divide by number of paths to these nodes: $(1+c).*t_3./q = w$
- Find 2nd neighbors: $Aw$
- Multiply by paths into 2nd neighbors and tally: $\tilde{c}+=Aw.*(q.*t_2)$

# Backward sweep of BC in GraphBLAS C API

```
…
  GrB_Monoid FP32Add;                                       // Monoid <float,+,0.0>
  GrB_Monoid_new(&FP32Add,GrB_FP32,GrB_PLUS_FP32,0.0f);
  GrB_Monoid FP32Mul;                                       // Monoid <float,*,1.0>
  GrB_Monoid_new(&FP32Mul,GrB_FP32,GrB_TIMES_FP32,1.0f);
  GrB_Semiring FP32AddMul;                                // Semiring <float,float,float,+,*,0.0>
  GrB_Semiring_new(&FP32AddMul,FP32Add,GrB_TIMES_FP32);

  GrB_Matrix nspinv;   // inverse of the number of shortest paths
  GrB_Matrix_new(&nspinv,GrB_FP32,n,nsver);
  GrB_apply(&nspinv,GrB_NULL,GrB_NULL,GrB_MINV_FP32,numsp,GrB_NULL);  // nspinv = 1./numsp

  GrB_Matrix bcu;   // BC updates for each starting vertex in s
  GrB_Matrix_new(&bcu,GrB_FP32,n,nsver);
  GrB_assign(&bcu,GrB_NULL,GrB_NULL,1.0f,GrB_ALL,n, GrB_ALL,nsver,GrB_NULL);
  // bcu is filled with 1 to avoid sparsity issues

  GrB_Descriptor desc_r;                          // Descriptor for 1st ewisemult in tally
  GrB_Descriptor_new(&desc_r);
  GrB_Descriptor_set(desc_r,GrB_OUTP,GrB_REPLACE);
  // clear output before result is stored in it.

  GrB_Matrix w;                                             // temporary workspace matrix
  GrB_Matrix_new(&w,GrB_FP32,n,nsver);
…
```

# Backward sweep of BC in GraphBLAS C API

```
…
  for (int i=d-1; i>0; i--)
  {  // ------------------- Tally phase (backward sweep) -------------------

    GrB_eWiseMult(&w,sigmas[i],GrB_NULL,FP32Mul,bcu,nspinv,desc_r);
    // w<sigmas[i]>=(1 ./ nsp).*bcu

    // add contributions by successors and mask with that BFS level's frontier
    GrB_mxm(&w,sigmas[i-1],GrB_NULL,FP32AddMul,A,w,desc_r);   // w<sigmas[i-1]> = (A +.* w)

    GrB_eWiseMult(&bcu,GrB_NULL,GrB_PLUS_FP32,FP32Mul,w,numsp,GrB_NULL);
    // bcu += w .* numsp
  }
  // subtract "nsver" from every entry in delta (1 extra value per bcu element crept in)
  GrB_assign(delta,GrB_NULL,GrB_NULL,-(float)nsver,GrB_ALL,n,GrB_NULL);   // fill with -nsver
  GrB_reduce(delta,GrB_NULL,GrB_PLUS_FP32,GrB_PLUS_FP32,bcu,GrB_NULL);
  // add all updates to -nsver

  for(int i=0; i<d; i++) { GrB_free(sigmas[i]); }
  free(sigmas);
  GrB_free_all(frontier,numsp,nspinv,w,bcu,desc_tsr,desc_r);
  // macro that expands GrB_free() for each parameter
  GrB_free_all(Int32AddMul,Int32Add,FP32AddMul,FP32Add,FP32Mul);
  return GrB_SUCCESS;
}
```

# Backward sweep of BC in GraphBLAS C API

```
…
  for (int i=d-1; i>0; i--)
  {  // ------------------- Tally phase (backward sweep) --------------------

    GrB_eWiseMult(&w,sigmas[i],GrB_NULL,FP32Mul,bcu,nspinv,desc_r);
    // w<sigmas[i]>=(1 ./ nsp).*bcu

    // add contributions by successors and mask with that BFS level's frontier
    GrB_mxm(&w,sigmas[i-1],GrB_NULL,FP32AddMul,A,w,desc_r);   // w<sigmas[i-1]> = (A +.* w)

    GrB_eW
    // bcu
  }
  // subtr                                                                  )
  GrB_assi                                                              -nsver
  GrB_redu
  // add a

  for(int i=0; i<d; i++) { GrB_free(sigmas[i]); }
  free(sigmas);
  GrB_free_all(frontier,numsp,nspinv,w,bcu,desc_tsr,desc_r);
  // macro that expands GrB_free() for each parameter
  GrB_free_all(Int32AddMul,Int32Add,FP32AddMul,FP32Add,FP32Mul);
  return GrB_SUCCESS;
}
```

- The contributions of each “end” vertex to its predecessors are divided by the number of shortest paths that reach them.
- This is accomplished with an eWiseMult operation where the sigma[i] matrix is used as a mask to ensure that only paths identified in the BFS phase (i.e. edges that belong to the BFS tree) are assigned to the result.

# Backward sweep of BC in GraphBLAS C API

```
…
  for (int i=d-1; i>0; i--)
  {  // ------------------- Tally phase (backward sweep) --------------------

    GrB_eWiseMult(&w,sigmas[i],GrB_NULL,FP32Mul,bcu,nspinv,desc_r);
    // w<sigmas[i]>=(1 ./ nsp).*bcu

    // add contributions by successors and mask with that BFS level's frontier
    GrB_mxm(&w,sigmas[i-1],GrB_NULL,FP32AddMul,A,w,desc_r);   // w<sigmas[i-1]> = (A +.* w)

    GrB_eWiseMult(&bcu,GrB_NULL,GrB_PLUS_FP32,FP32Mul,w,numsp,GrB_NULL);
    // bcu += w .* numsp
  }
  // subtract "nsver" from every entry in delta (1 extra value per bcu element crept in)
  GrB_assign(de
  GrB_reduce(de
  // add all up

  for(int i=0;
  free(sigmas);
  GrB_free_all(
  // macro that
  GrB_free_all(
  return GrB_SU
}
```

- The GrB_mxm call discovers *predecessors* (as opposed to successors in the forward sweep) by its use of the descriptor desc_r that uses the adjacency matrix (as opposed to its transpose).
- The algorithm assures that the BC contributions are transferred only to direct parents on the BFS tree by passing the previous level of BFS tree (sigma[i-1]) as a mask to GrB_mxm.