Supplementary Material for Linearized Alternating Direction Method with Adaptive Penalty for Low-Rank Representation

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1 Proof of Proposition 2

We present the proof of Proposition 2 in a more general setting. Namely, λ is updated as

$$\lambda_{k+1} = \lambda_k + \gamma\beta_k[\mathcal{A}(\mathbf{x}_{k+1}) + \mathcal{B}(\mathbf{y}_{k+1}) - \mathbf{c}]. \tag{21}$$

Even with this extra parameter γ, the proof of Theorem 3 is almost unchanged. We have a more general Proposition 2 as follows:

Proposition 2 *If $\{\beta_k\}$ is non-decreasing and upper bounded, $\eta_A > \|\mathcal{A}\|^2$, $\gamma \in (0, 2)$, $\eta_B(2-\gamma) > \|\mathcal{B}\|^2$, and $(\mathbf{x}^*, \mathbf{y}^*, \lambda^*)$ is any KKT point of problem (1), then:*

1. $\{\eta_A\|\mathbf{x}_k - \mathbf{x}^*\|^2 - \|\mathcal{A}(\mathbf{x}_k - \mathbf{x}^*)\|^2 + \eta_B\|\mathbf{y}_k - \mathbf{y}^*\|^2 + \gamma^{-1}\beta_k^{-2}\|\lambda_k - \lambda^*\|^2\}$ *is non-increasing.*
2. $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \to 0$, $\|\mathbf{y}_{k+1} - \mathbf{y}_k\| \to 0$, $\|\lambda_{k+1} - \lambda_k\| \to 0$.

The proof of Proposition 2 is based on the following lemma.

Lemma 1

$$\begin{array}{l}
\eta_A\|\mathbf{x}_{k+1} - \mathbf{x}^*\|^2 - \|\mathcal{A}(\mathbf{x}_{k+1} - \mathbf{x}^*)\|^2 + \eta_B\|\mathbf{y}_{k+1} - \mathbf{y}^*\|^2 + \gamma^{-1}\beta_k^{-2}\|\lambda_{k+1} - \lambda^*\|^2 \\
= \eta_A\|\mathbf{x}_k - \mathbf{x}^*\|^2 - \|\mathcal{A}(\mathbf{x}_k - \mathbf{x}^*)\|^2 + \eta_B\|\mathbf{y}_k - \mathbf{y}^*\|^2 + \gamma^{-1}\beta_k^{-2}\|\lambda_k - \lambda^*\|^2 \\
- \big\{(2-\gamma)(\gamma\beta_k)^{-2}\|\lambda_{k+1} - \lambda_k\|^2 + \eta_B\|\mathbf{y}_{k+1} - \mathbf{y}_k\|^2 \\
\quad -2(\gamma\beta_k)^{-1}\langle\lambda_{k+1} - \lambda_k, \mathcal{B}(\mathbf{y}_{k+1} - \mathbf{y}_k)\rangle\big\} \\
-(\eta_A\|\mathbf{x}_{k+1} - \mathbf{x}_k\|^2 - \|\mathcal{A}(\mathbf{x}_{k+1} - \mathbf{x}_k)\|^2) \\
-2\beta_k^{-1}\left\langle \mathbf{x}_{k+1} - \mathbf{x}^*, [-\beta_k\eta_A(\mathbf{x}_{k+1} - \mathbf{x}_k) - \mathcal{A}^*(\tilde{\lambda}_{k+1})] + \mathcal{A}^*(\lambda^*)\right\rangle \\
-2\beta_k^{-1}\left\langle \mathbf{y}_{k+1} - \mathbf{y}^*, [-\beta_k\eta_B(\mathbf{y}_{k+1} - \mathbf{y}_k) - \mathcal{B}^*(\hat{\lambda}_{k+1})] + \mathcal{B}^*(\lambda^*)\right\rangle.
\end{array} \tag{22}$$

This identity can be routinely checked, by using the definitions of $\tilde{\lambda}_{k+1}$ and $\hat{\lambda}_{k+1}$ and the following facts:

1. $2\langle\mathbf{a}_{k+1} - \mathbf{a}^*, \mathbf{a}_{k+1} - \mathbf{a}_k\rangle = \|\mathbf{a}_{k+1} - \mathbf{a}^*\|^2 - \|\mathbf{a}_k - \mathbf{a}^*\|^2 + \|\mathbf{a}_{k+1} - \mathbf{a}_k\|^2$.
2. $\mathcal{A}(\mathbf{x}^*) + \mathcal{B}(\mathbf{y}^*) = \mathbf{c}$.
3. $\langle\lambda, \mathcal{A}(\mathbf{x})\rangle = \langle\mathcal{A}^*(\lambda), \mathbf{x}\rangle$, $\langle\lambda, \mathcal{B}(\mathbf{y})\rangle = \langle\mathcal{B}^*(\lambda), \mathbf{y}\rangle$.

As it is lengthy and tedious, we omit the complete details.

Proof (of Proposition 2) By Lemma 1 and the given conditions, it is easy to check that

$$\eta_A\|\mathbf{w}\|^2 - \|\mathcal{A}(\mathbf{w})\|^2 \geq 0, \quad \text{for } \mathbf{w} = \mathbf{x}_{k+1} - \mathbf{x}^*, \mathbf{x}_k - \mathbf{x}^*, \mathbf{x}_{k+1} - \mathbf{x}_k,$$

$$(2-\gamma)(\gamma\beta_k)^{-2}\|\lambda_{k+1}-\lambda_k\|^2+\eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-2(\gamma\beta_k)^{-1}\langle\lambda_{k+1}-\lambda_k,\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\rangle\geq 0.$$

The last two terms in (22) are also nonnegative due to Proposition 1 and the monotonicity of subgradient mapping. So Proposition 2 (1) is obvious due to the non-decrement of $\{\beta_k\}$.

Then as $\{\eta_A\|\mathbf{x}_k-\mathbf{x}^*\|^2-\|\mathcal{A}(\mathbf{x}_k-\mathbf{x}^*)\|^2+\eta_B\|\mathbf{x}_k-\mathbf{x}^*\|^2+\gamma^{-1}\beta_k^{-2}\|\lambda_k-\lambda^*\|^2\}$ is non-increasing and non-negative, it has a limit. Then we can see that

$$\eta_A\|\mathbf{x}_{k+1}-\mathbf{x}_k\|^2-\|\mathcal{A}(\mathbf{x}_{k+1}-\mathbf{x}_k)\|^2\to 0,$$

$$(2-\gamma)(\gamma\beta_k)^{-2}\|\lambda_{k+1}-\lambda_k\|^2+\eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-2(\gamma\beta_k)^{-1}\langle\lambda_{k+1}-\lambda_k,\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\rangle\to 0,$$

due to their non-negativity. So $\|\mathbf{x}_{k+1}-\mathbf{x}_k\|\to 0$ follows from the first limit.

Note that

$$\begin{aligned}
&(2-\gamma)(\gamma\beta_k)^{-2}\|\lambda_{k+1}-\lambda_k\|^2+\eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-2(\gamma\beta_k)^{-1}\langle\lambda_{k+1}-\lambda_k,\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\rangle\\
&\geq (2-\gamma)(\gamma\beta_k)^{-2}\|\lambda_{k+1}-\lambda_k\|^2+\eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-2(\gamma\beta_k)^{-1}\|\lambda_{k+1}-\lambda_k\|\|\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\|\\
&=\left((2-\gamma)^{1/2}(\gamma\beta_k)^{-1}\|\lambda_{k+1}-\lambda_k\|-(2-\gamma)^{-1/2}\|\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\|\right)^2\\
&+\eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-(2-\gamma)^{-1}\|\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\|^2\\
&\geq \eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-(2-\gamma)^{-1}\|\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\|^2.
\end{aligned}$$

So we have that $\|\mathbf{y}_{k+1}-\mathbf{y}_k\|\to 0$. On the other hand,

$$\begin{aligned}
&(2-\gamma)(\gamma\beta_k)^{-2}\|\lambda_{k+1}-\lambda_k\|^2+\eta_B\|\mathbf{y}_{k+1}-\mathbf{y}_k\|^2-2(\gamma\beta_k)^{-1}\langle\lambda_{k+1}-\lambda_k,\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\rangle\\
&=\left((2-\gamma)^{1/2}(\gamma\beta_k)^{-1}\|\lambda_{k+1}-\lambda_k\|-\sqrt{\eta_B}\|\mathbf{y}_{k+1}-\mathbf{y}_k\|\right)^2\\
&+2(\gamma\beta_k)^{-1}\left(\sqrt{\eta_B(2-\gamma)}\|\lambda_{k+1}-\lambda_k\|\|\mathbf{y}_{k+1}-\mathbf{y}_k\|-\langle\lambda_{k+1}-\lambda_k,\mathcal{B}(\mathbf{y}_{k+1}-\mathbf{y}_k)\rangle\right)\\
&\geq\left((2-\gamma)^{1/2}(\gamma\beta_k)^{-1}\|\lambda_{k+1}-\lambda_k\|-\sqrt{\eta_B}\|\mathbf{y}_{k+1}-\mathbf{y}_k\|\right)^2.
\end{aligned}$$

So $(2-\gamma)^{1/2}(\gamma\beta_k)^{-1}\|\lambda_{k+1}-\lambda_k\|-\sqrt{\eta_B}\|\mathbf{y}_{k+1}-\mathbf{y}_k\|\to 0$. This together with $\|\mathbf{y}_{k+1}-\mathbf{y}_k\|\to 0$ results in $\|\lambda_{k+1}-\lambda_k\|\to 0$.

2 Solving LRR via APG

The LRR problem can also be relaxed to the following unconstrained optimization problem:

$$\min \beta\|\mathbf{Z}\|_*+\beta\mu\|\mathbf{E}\|_{2,1}+\frac{1}{2}\|\mathbf{X}-\mathbf{XZ}-\mathbf{E}\|^2, \tag{23}$$

where $\beta>0$ is a relaxation parameter. Then we can apply APG to solve this problem. The two subproblems to update $\mathbf{E}$ and $\mathbf{Z}$ are:

$$\mathbf{E}_{k+1}=\arg\min_{\mathbf{E}}\mu\beta\|\mathbf{E}\|_{2,1}+\frac{\tau}{2}\|\mathbf{E}-(\bar{\mathbf{E}}_k-\frac{1}{2\tau}\nabla_E\|\mathbf{X}-\mathbf{XZ}-\mathbf{E}\|^2|_{\bar{\mathbf{E}}_k,\bar{\mathbf{Z}}_k})\|^2, \tag{24a}$$

$$\mathbf{Z}_{k+1}=\arg\min_{\mathbf{Z}}\beta\|\mathbf{Z}\|_*+\frac{\tau}{2}\|\mathbf{Z}-(\bar{\mathbf{Z}}_k-\frac{1}{2\tau}\nabla_Z\|\mathbf{X}-\mathbf{XZ}-\mathbf{E}\|^2|_{\bar{\mathbf{E}}_k,\bar{\mathbf{Z}}_k})\|^2, \tag{24b}$$

where $\tau\geq\sigma_{\max}^2(\mathbf{X})$ is a Lipschitz constant.

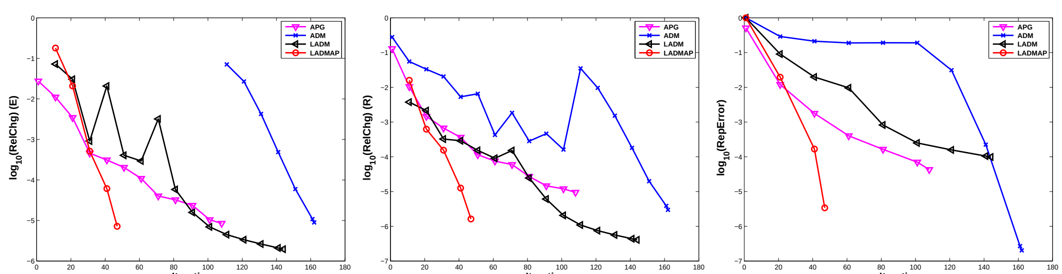
The APG approach, with the continuation technique, for the LRR problem is described in Algorithm 3.

3 Convergence Behaviors of Tested Algorithms

In Figure 1, we plot the relative changes of $\mathbf{E}_k$ and $\mathbf{Z}_k$ and the feasibility errors at all iterations for four test algorithms, respectively. We can see the errors of LADMAP in the two KKT conditions drop much quicker than other methods.

Algorithm 3 APG for LRR

Input: Observation matrix $\mathbf{X}$ and parameter $\mu > 0$.
Initialize: Set $\mathbf{E}_0 = \mathbf{E}_{-1} = \mathbf{0}$ and $\mathbf{Z}_0 = \mathbf{Z}_{-1} = \mathbf{0}$.
Set $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\beta_0 \gg \beta_{\min} > 0$, $t_0 = t_{-1} = 1$, $\theta < 1$, $\tau \geq \sigma_{\max}^2(\mathbf{X})$, and $k \leftarrow 0$.
while not converged **do**
Step 1: Update $\bar{\mathbf{E}}_k = \mathbf{E}_k + \frac{t_{k-1}-1}{t_k}(\mathbf{E}_k - \mathbf{E}_{k-1})$, $\bar{\mathbf{Z}}_k = \mathbf{Z}_k + \frac{t_{k-1}-1}{t_k}(\mathbf{Z}_k - \mathbf{Z}_{k-1})$.
Step 2: Update $\mathbf{G}_k^E = \bar{\mathbf{E}}_k + \frac{1}{\tau}(\mathbf{X} - \mathbf{X}\bar{\mathbf{Z}}_k - \bar{\mathbf{E}}_k)$.
Step 3: Update $\mathbf{E}_{k+1} = \mathcal{S}_{\frac{\mu\beta_k}{\tau}}(\mathbf{G}_k^E)$, where $\mathcal{S}$ is the shrinkage operator.
Step 4: Update $\mathbf{G}_k^Z = \bar{\mathbf{Z}}_k + \frac{1}{\tau}\mathbf{X}^T(\mathbf{X} - \mathbf{X}\bar{\mathbf{Z}}_k - \bar{\mathbf{E}}_k)$.
Step 5: Update $\mathbf{Z}_{k+1} = \mathbf{U}\mathcal{S}_{\frac{\beta_k}{\tau}}(\Sigma)\mathbf{V}^T$, where $\mathbf{U}\Sigma\mathbf{V}^T$ is the SVD of $\mathbf{G}_k^Z$.
Step 6: Update $t_{k+1} = \frac{1+\sqrt{4t_k^2+1}}{2}$, $\beta_{k+1} = \max(\beta_{\min}, \theta\beta_k)$.
Step 7: Check the convergence conditions:
$\frac{\|\mathbf{X}\mathbf{Z}_{k+1}+\mathbf{E}_{k+1}-\mathbf{X}\|}{\|\mathbf{X}\|} \leq \varepsilon_1$ and $\max\left(\frac{\|\mathbf{Z}_{k+1}-\mathbf{Z}_k\|}{\|\mathbf{X}\|}, \frac{\|\mathbf{E}_{k+1}-\mathbf{E}_k\|}{\|\mathbf{X}\|}\right) \leq \varepsilon_2$.
If they are satisfied, break.
Step 8: $k \leftarrow k + 1$.
end while



(a) Relative Changes of $\mathbf{E}_k$ (b) Relative Changes of $\mathbf{Z}_k$ (c) Feasibility Errors

Figure 1: Convergence behaviors of APG, ADM, LADM, LADMAP on the toy data $\mathbf{X}$ generated with parameters (5, 20, 100, 5). The changes and errors are in $\log_{10}$ scale. In (a) and (b), as the relative changes of $\mathbf{E}_k$ and $\mathbf{Z}_k$ in the first several iterations are zeros, which corresponds to $-\infty$ in the plots, we only report the nonzero relative changes of $\mathbf{E}_k$ and $\mathbf{Z}_k$.